

The Paths of Gravity in Galileon Cosmology

Stephen Appleby¹ and Eric V. Linder^{1,2}

¹ Institute for the Early Universe WCU, Ewha Womans University, Seoul, Korea and

² Berkeley Lab & University of California, Berkeley, CA 94720, USA

(Dated: April 20, 2012)

Galileon gravity offers a robust gravitational theory for explaining cosmic acceleration, having a rich phenomenology of testable behaviors. We explore three classes of Galileon models – standard uncoupled, and linearly or derivatively coupled to matter – investigating the expansion history with particular attention to early time and late time attractors, as well as the linear perturbations. From the relativistic and nonrelativistic Poisson equations we calculate the generalizations of the gravitational strength (Newton’s constant), deriving its early and late time behavior. By scanning through the parameters we derive distributions of the gravitational strength at various epochs and trace the paths of gravity in its evolution. Using ghost-free and stability criteria we restrict the allowed parameter space, finding in particular that the linear and derivative coupled models are severely constrained by classical instabilities in the early universe.

I. INTRODUCTION

The origin of the current cosmic acceleration is a fundamental mystery, even to whether it arises from a new field or a change in the laws of gravity. Scalar-tensor gravity is an intriguing possibility, partaking of elements of both classes of explanation. Those theories that do not have a scalar potential avoid some naturalness problems associated with this, and those which involve a symmetry or geometric origin moreover sidestep difficulties with high energy physics corrections.

A class of theories possessing all these desiderata is Galileon gravity [1–4], involving a shift symmetry in the field, while the field itself can be viewed as a geometric object arising from higher dimensions and entering in invariant combinations that assure second order field equations and protect against various pathologies. The derivative self couplings of the field also screen deviations from general relativity in high gradient regions (e.g. small scales or high densities), thus satisfying solar system and early universe constraints [5–7]. The galileon field, and closely related models [8–16], have been extensively studied in both theoretical [17–25] and observational [26–33] contexts (and also with respect to inflation; [34–36]).

Galileon gravity thus merits investigation in detail, from both the expansion properties (homogeneous background) and structure growth (inhomogeneous perturbations) perspectives. Regarding the expansion, we will particularly be interested in the early universe corrections to radiation and matter domination, the effective equation of state evolution, and late time attractor behavior, especially to a de Sitter state. For the behavior of linearized perturbations (concentrating on subhorizon scales), we explore the form of the modified gravitational strengths (Newton constants) in the generalized Poisson equations, looking for characteristic features in their evolution.

The parameter space of the theory will be constrained by general physical considerations, such as positivity of the effective energy density and freedom from ghosts (Hamiltonian bounded from below). Observationally, we ask that the radiation and matter eras remain intact and the effective dark energy density is of order 3/4 the critical density today. These conditions still permit considerable diversity in the gravitational behavior, which is quite interesting to explore.

In Section II we briefly review the standard Galileon model and include terms with linear and derivative coupling to matter, setting up the field equations and modified Friedmann equations for a homogeneous, isotropic universe. Section III sets up the linear perturbation theory and the generalized Poisson equations, deriving expressions for the gravitational strength modifications. Solutions for the expansion history for the uncoupled, linear-, and derivative-coupled models are presented in Section IV, including discussion of early and late time attractors. Evolution of the gravitational strengths is studied in Section V, along with the no-ghost and stability conditions. We summarize the results and conclude in Section VI.

II. CLASSES OF GALILEON GRAVITY

Galileon gravity is a scalar field theory containing nonlinear derivative self couplings. The action for the scalar field π is invariant under Galilean symmetries $\pi \rightarrow \pi + c + b_\mu x^\mu$ in the absence of gravity, where c and b_μ are a constant scalar and four vector respectively. The four dimensional action that preserves these symmetries contains five unique terms, consisting of scalar combinations of $\partial_\mu \pi$, $\partial_\mu \partial_\nu \pi$ and $\square \pi$.

These models share many similarities with the decoupling limit of DGP gravity [37–39] and exhibit Vainshtein screening that restores the theory to the general relativity behavior under certain conditions. Galileon gravity has field equations that contain at most second derivatives of the scalar field, and hence can be free of ghost instabilities. In addition, since these second derivative terms appear linearly, the Cauchy problem can be well defined.

When gravity is introduced one must necessarily break the Galilean $\partial_\mu \pi \rightarrow \partial_\mu \pi + b_\mu$ symmetry. However, a covariantized formulation of the Galileon model has been constructed in [2], in which the action preserves the shift symmetry $\pi \rightarrow \pi + c$, and the Galilean symmetry is softly broken.

The covariant Galileon action can be written as

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{pl}}^2 R}{2} - \frac{1}{2} \sum_{i=1}^5 c_i \mathcal{L}_i - \mathcal{L}_{\text{m}} \right] \quad (1)$$

where c_{1-5} are arbitrary dimensionless constants, g is the determinant of the metric, M_{pl} is the Planck mass, and R is the Ricci scalar. The Galileon Lagrangians are given by

$$\mathcal{L}_1 = M^3 \pi, \quad \mathcal{L}_2 = (\nabla_\mu \pi)(\nabla^\mu \pi), \quad \mathcal{L}_3 = (\square \pi)(\nabla_\mu \pi)(\nabla^\mu \pi)/M^3 \quad (2)$$

$$\mathcal{L}_4 = (\nabla_\mu \pi)(\nabla^\mu \pi) \left[2(\square \pi)^2 - 2\pi_{;\mu\nu} \pi^{;\mu\nu} - R(\nabla_\mu \pi)(\nabla^\mu \pi)/2 \right] /M^6 \quad (3)$$

$$\mathcal{L}_5 = (\nabla_\mu \pi)(\nabla^\mu \pi) \left[(\square \pi)^3 - 3(\square \pi)\pi_{;\mu\nu} \pi^{;\mu\nu} + 2\pi_{;\mu}^{;\nu} \pi_{;\nu}^{;\rho} \pi_{;\rho}^{;\mu} - 6\pi_{;\mu} \pi^{;\mu\nu} \pi^{;\rho} G_{\nu\rho} \right] /M^9 \quad (4)$$

where M is the mass dimension taken out of the couplings to make the c 's dimensionless; without loss of generality we can take $M^3 = M_{\text{pl}} H_0^2$.

The Galileon invariance severely restricts the form of the action, although further freedom remains in the coupling between π and matter. For example, the weak field limit of higher dimensional brane models typically involve a coupling of the form πT , where T is the trace of the energy-momentum tensor; such a term explicitly enters into the effective scalar field action in the decoupling limit of the DGP model [43]. We also consider a derivative coupling $T^{\mu\nu} \partial_\mu \pi \partial_\nu \pi$, which has other interesting properties and origins. We will set the tadpole term to zero, i.e. $c_1 = 0$, since we are interested in the effect of derivative self couplings on the growth and expansion histories and wish to avoid an explicit cosmological constant.

Thus, the action can be written as

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{pl}}^2 R}{2} - \frac{c_2}{2} (\partial \pi)^2 - \frac{c_3}{M^3} (\partial \pi)^2 \square \pi - \frac{c_4 \mathcal{L}_4}{2} - \frac{c_5 \mathcal{L}_5}{2} - \mathcal{L}_{\text{m}} - \frac{c_G}{M_{\text{pl}} M^3} T^{\mu\nu} \partial_\mu \pi \partial_\nu \pi - \frac{c_0}{M_{\text{pl}}} \pi T \right]. \quad (5)$$

For computational purposes we find it is easier to work in the Jordan frame, where the explicit coupling between π and matter is removed via a metric redefinition. In Appendix A we perform these metric transformations in the weak field limit, absorb like terms by renormalizing the c 's, and promote the final actions to their full, non-linear counterparts. Here we state the result:

$$S = \int d^4x \sqrt{-g} \left[\left(1 - 2c_0 \frac{\pi}{M_{\text{pl}}} \right) \frac{M_{\text{pl}}^2 R}{2} - \frac{c_2}{2} (\partial \pi)^2 - \frac{c_3}{M^3} (\partial \pi)^2 \square \pi - \frac{c_4 \mathcal{L}_4}{2} - \frac{c_5 \mathcal{L}_5}{2} - \frac{M_{\text{pl}}}{M^3} c_G G^{\mu\nu} \partial_\mu \pi \partial_\nu \pi - \mathcal{L}_{\text{m}} \right]. \quad (6)$$

Varying the action with respect to the fields $g_{\mu\nu}$ and π , we obtain the general field equations that are exhibited in Appendix B.

In what follows we analyse three classes of models: the standard uncoupled Galileon, where $c_0 = c_G = 0$, the linearly coupled Galileon, where $c_G = 0$, and the derivative coupled Galileon, where $c_0 = 0$. Note that derivative coupling resulting in such $G^{\mu\nu} \partial_\mu \pi \partial_\nu \pi$ terms has been found of interest for dark energy in [44–46] and inflation in [47], and has ties to higher dimensional, vector, and disformal gravity theories.

To derive the cosmological evolution of these classes of Galileon gravity, we use the Robertson-Walker metric for a homogeneous, isotropic spacetime,

$$ds^2 = -dt^2 + a^2 \delta_{ij} dx^i dx^j, \quad (7)$$

where a is the cosmic scale factor and we assume spatial flatness for simplicity. For the homogeneous evolution $\pi = \pi(a)$ and there are two independent field equations.

We can write the (i, i) Einstein equation and π dynamical equation in terms of dimensionless variables $y = \pi/M_{\text{pl}}$, $\bar{H} = H/H_0$, and $x = \pi'/M_{\text{pl}}$, where the Hubble parameter $H = \dot{a}/a$ and a prime denotes $d/d \ln a$, as a set of dynamical

equations

$$x' = -x + \frac{\alpha\lambda - \sigma\gamma}{\sigma\beta - \alpha\omega} \quad (8)$$

$$\bar{H}' = -\frac{\lambda}{\sigma} + \frac{\omega}{\sigma} \left(\frac{\sigma\gamma - \alpha\lambda}{\sigma\beta - \alpha\omega} \right) \quad (9)$$

$$y' = x \quad (10)$$

where

$$\alpha = -3c_3\bar{H}^3x^2 + 15c_4\bar{H}^5x^3 + c_0\bar{H} + \frac{c_2\bar{H}x}{6} - \frac{35}{2}c_5\bar{H}^7x^4 - 3c_G\bar{H}^3x \quad (11)$$

$$\gamma = 2c_0\bar{H}^2 - c_3\bar{H}^4x^2 + \frac{c_2\bar{H}^2x}{3} + \frac{5}{2}c_5\bar{H}^8x^4 - 2c_G\bar{H}^4x \quad (12)$$

$$\beta = -2c_3\bar{H}^4x + \frac{c_2\bar{H}^2}{6} + 9c_4\bar{H}^6x^2 - 10c_5\bar{H}^8x^3 - c_G\bar{H}^4 \quad (13)$$

$$\sigma = 2(1 - 2c_0y)\bar{H} - 2c_0\bar{H}x + 2c_3\bar{H}^3x^3 - 15c_4\bar{H}^5x^4 + 21c_5\bar{H}^7x^5 + 6c_G\bar{H}^3x^2 \quad (14)$$

$$\begin{aligned} \lambda &= 3(1 - 2c_0y)\bar{H}^2 - 2c_0\bar{H}^2x \\ &\quad - 2c_3\bar{H}^4x^3 + \frac{c_2\bar{H}^2x^2}{2} + \frac{\Omega_{r0}}{a^4} + \frac{15}{2}c_4\bar{H}^6x^4 - 9c_5\bar{H}^8x^5 - c_G\bar{H}^4x^2 \end{aligned} \quad (15)$$

$$\omega = -2c_0\bar{H}^2 + 2c_3\bar{H}^4x^2 - 12c_4\bar{H}^6x^3 + 15c_5\bar{H}^8x^4 + 4c_G\bar{H}^4x. \quad (16)$$

As a check of our numerical solutions we verify that the redundant Friedmann equation is satisfied at all times during the evolution:

$$(1 - 2c_0y)\bar{H}^2 = \frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4} + 2c_0\bar{H}^2x + \frac{c_2}{6}\bar{H}^2x^2 - 2c_3\bar{H}^4x^3 + \frac{15}{2}c_4\bar{H}^6x^4 - 7c_5\bar{H}^8x^5 - 3c_G\bar{H}^4x^2, \quad (17)$$

where Ω_{m0} and Ω_{r0} are the present matter and radiation energy densities, respectively, in units of the critical density.

It is also useful to write the energy density and pressure of the scalar field:

$$\frac{\rho_\pi}{H_0^2 M_{\text{pl}}^2} = 6c_0\bar{H}^2x + \frac{c_2}{2}\bar{H}^2x^2 - 6c_3\bar{H}^4x^3 + \frac{45}{2}c_4\bar{H}^6x^4 - 21c_5\bar{H}^8x^5 - 9c_G\bar{H}^4x^2 \quad (18)$$

$$\begin{aligned} \frac{P_\pi}{H_0^2 M_{\text{pl}}^2} &= -c_0 \left[4\bar{H}^2x + 2\bar{H}(\bar{H}x)' \right] + \frac{c_2}{2}\bar{H}^2x^2 + 2c_3\bar{H}^3x^2(\bar{H}x)' - c_4 \left[\frac{9}{2}\bar{H}^6x^4 + 12\bar{H}^6x^3x' + 15\bar{H}^5x^4\bar{H}' \right] \\ &\quad + 3c_5\bar{H}^7x^4(5\bar{H}x' + 7\bar{H}'x + 2\bar{H}x) + c_G[6\bar{H}^3x^2\bar{H}' + 4\bar{H}^4xx' + 3\bar{H}^4x^2] \end{aligned} \quad (19)$$

from which one can define an effective equation of state parameter for the Galileon: $w \equiv P_\pi/\rho_\pi$.

III. INHOMOGENEOUS PERTURBATIONS AND MODIFIED POISSON EQUATIONS

While the previous section set up the background dynamics for the cosmological evolution, we also need the inhomogeneous equations to investigate the effects of perturbations on the cosmic growth history and gravitational modifications to the generalized Poisson equations. We here consider linear perturbation theory for scalar modes in the subhorizon limit. In the Newtonian gauge the perturbed metric is

$$ds^2 = -(1 + 2\psi)dt^2 + a^2(1 - 2\phi)\delta_{ij}dx^i dx^j. \quad (20)$$

The linearized Einstein and scalar field equations are exhibited in Appendix C. These lead to the modified Poisson equations

$$\bar{\nabla}^2\phi = \frac{4\pi a^2 G_{\text{eff}}^{(\phi)} \rho_m}{H_0^2} \delta_m \quad (21)$$

$$\bar{\nabla}^2\psi = \frac{4\pi a^2 G_{\text{eff}}^{(\psi)} \rho_m}{H_0^2} \delta_m \quad (22)$$

$$\bar{\nabla}^2(\psi + \phi) = \frac{8\pi a^2 G_{\text{eff}}^{(\psi+\phi)} \rho_m}{H_0^2} \delta_m, \quad (23)$$

where we describe the effect of the Galileon on subhorizon density perturbations through modifications G_{eff} of the gravitational strength, or effective Newton's constant. These will be functions of the background quantities \bar{H} , y , and their derivatives.

Note that $G_{\text{eff}}^{(\psi)}$ is central to the growth of cosmic structure and is called \mathcal{V} in [48] and μ in [49], while $G_{\text{eff}}^{(\psi+\phi)}$ is important for light deflection and the integrated Sachs-Wolfe effect, and called \mathcal{G} and Σ in those two references, respectively. Translation tables between other parametrizations are given in [48, 50]. One can regard the ψ Poisson equation as governing nonrelativistic geodesics and the $\psi + \phi$ Poisson equation as governing relativistic geodesics in the inhomogeneous spacetime [51, 52].

The modified gravitational strengths are given in terms of Newton's constant G_N by

$$G_{\text{eff}}^{(\phi)} = \frac{2(\kappa_4\kappa_6 - \kappa_5\kappa_1)}{\kappa_5(\kappa_4\kappa_1 - \kappa_5\kappa_3) - \kappa_4(\kappa_4\kappa_6 - \kappa_5\kappa_1)} G_N \quad (24)$$

$$G_{\text{eff}}^{(\psi)} = \frac{4(\kappa_3\kappa_6 - \kappa_1^2)}{\kappa_5(\kappa_4\kappa_1 - \kappa_5\kappa_3) - \kappa_4(\kappa_4\kappa_6 - \kappa_5\kappa_1)} G_N \quad (25)$$

$$G_{\text{eff}}^{(\psi+\phi)} = \frac{\kappa_6(2\kappa_3 + \kappa_4) - \kappa_1(2\kappa_1 + \kappa_5)}{\kappa_5(\kappa_4\kappa_1 - \kappa_5\kappa_3) - \kappa_4(\kappa_4\kappa_6 - \kappa_5\kappa_1)} G_N \quad (26)$$

where κ_{1-6} are functions of the background variables:

$$\begin{aligned} \kappa_1 &= -6c_4\bar{H}^3x^2 \left(\bar{H}'x + \bar{H}x' + \frac{\bar{H}x}{3} \right) + 2c_G(\bar{H}\bar{H}'x + \bar{H}^2x' + \bar{H}^2x) - 2c_0 + c_5\bar{H}^5x^3(12\bar{H}x' + 15\bar{H}'x + 3\bar{H}x) \\ \kappa_2 &= -\frac{c_2}{2} + 6c_3\bar{H}^2x + 3c_G\bar{H}^2 - 27c_4\bar{H}^4x^2 + 30c_5\bar{H}^6x^3 \\ \kappa_3 &= -(1 - 2c_0y) - \frac{c_4}{2}\bar{H}^4x^4 + c_G\bar{H}^2x^2 - 3c_5\bar{H}^5x^4(\bar{H}x' + \bar{H}'x) \\ \kappa_4 &= -2(1 - 2c_0y) + 3c_4\bar{H}^4x^4 - 2c_G\bar{H}^2x^2 - 6c_5\bar{H}^6x^5 \\ \kappa_5 &= 2c_3\bar{H}^2x^2 - 12c_4\bar{H}^4x^3 + 4c_G\bar{H}^2x - 2c_0 + 15c_5\bar{H}^6x^4 \\ \kappa_6 &= \frac{c_2}{2} - 2c_3(\bar{H}^2x' + \bar{H}\bar{H}'x + 2\bar{H}^2x) + c_4(12\bar{H}^4xx' + 18\bar{H}^3x^2\bar{H}' + 13\bar{H}^4x^2) \\ &\quad - c_G(2\bar{H}\bar{H}' + 3\bar{H}^2) - c_5(18\bar{H}^6x^2x' + 30\bar{H}^5x^3\bar{H}' + 12\bar{H}^6x^3) . \end{aligned}$$

The general relativistic form of the Poisson equation is recovered if we set $c_0 = c_{3-5} = c_G = 0$ (c_2 can remain unspecified; in this limit we have a massless, minimally coupled scalar field which will not modify the effective Newton constants). In that case, $G_{\text{eff}}^{(\psi)} = G_{\text{eff}}^{(\phi)} = G_{\text{eff}}^{(\psi+\phi)} = G_N$.

IV. EXPANSION HISTORY IN GALILEON COSMOLOGY

We now evaluate how Galileon cosmology affects the cosmic expansion history relative to the standard Λ CDM paradigm, including early universe asymptotes, the onset of cosmic acceleration, and late time attractors, especially to a de Sitter state.

A. Uncoupled Galileon: $c_0 = 0 = c_G$

The first, simplest case corresponds to the absence of any direct coupling between the scalar field π and matter; the parameters are then c_2, c_3, c_4, c_5 . Such a model has been extensively studied in [30–32]. Equations (8)–(10) describing the background evolution define an autonomous system, and it is straightforward to establish the existence of asymptotic fixed points.

The only fixed points for matter and radiation domination correspond to $(\Omega_m, \Omega_r, x) = (1, 0, 0)$ and $(0, 1, 0)$ respectively. We therefore solve the π equation assuming matter and radiation domination to obtain the behaviour of ρ_π at early times. For late times, we want cosmic acceleration and so we look for a late time de Sitter asymptotic state by finding the fixed points of the dynamical system (\bar{H}, x) , taking $\Omega_m = \Omega_r = 0$.

The full numerical solutions are illustrated in Figure 1 for specific parameter choices, and agree asymptotically with the fixed point behaviors. We plot the time evolution of \bar{H}^2 , $\rho_m/(3H_0^2M_{\text{pl}}^2)$, $\rho_r/(3H_0^2M_{\text{pl}}^2)$ and $\rho_\pi/(3H_0^2M_{\text{pl}}^2)$ in the left panel, and the effective dark energy equation of state parameter w in the right panel.

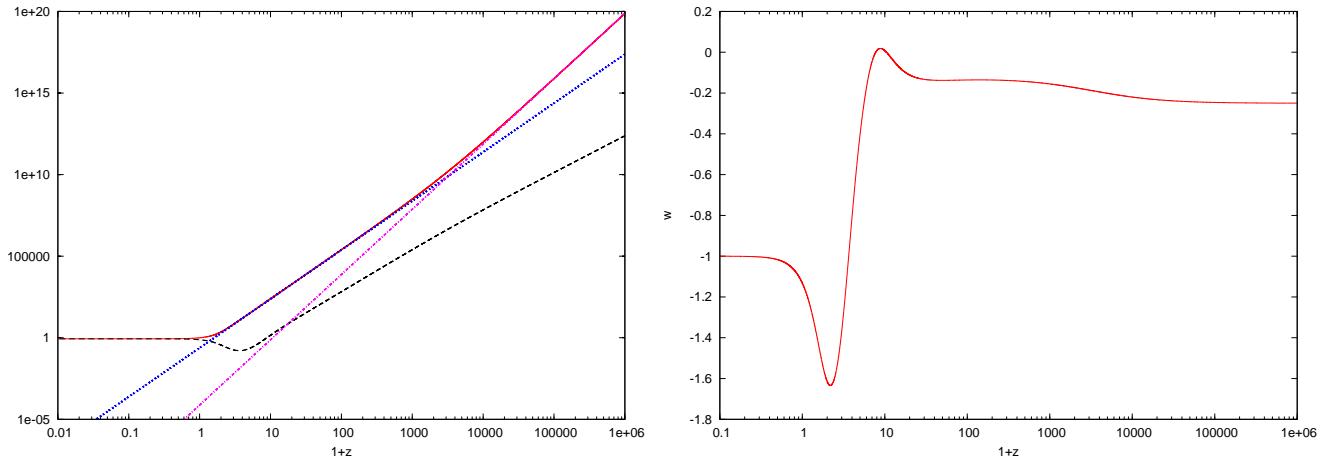


FIG. 1: [Left panel] The redshift evolution of \bar{H}^2 (solid red), $8\pi G\rho_m/3H_0^2$ (dotted blue), $8\pi G\rho_r/3H_0^2$ (dot-dashed magenta) and $8\pi G\rho_\pi/3H_0^2$ (long dash black) for the uncoupled Galileon, with parameters $c_2 = -27.3$, $c_3 = -12.8$, $c_4 = -1.7$, $c_5 = -1$. Here, the scalar field energy density slowly grows relative to matter at early times, as expected from the analytic results. The c_5 term dominates the π dynamics up to $z \sim 10$, after which we observe a turnaround to approach the late time de Sitter point. [Right panel] The effective equation of state of the Galileon field. During radiation domination $w = -1/4$, in agreement with the analytic results; the model currently accelerates the expansion and approaches a late time de Sitter point.

To calculate the behaviour of the π field at early times we look for solutions taking $\bar{H} \sim a^{-2}$ (during radiation domination) or $\bar{H} \sim a^{-3/2}$ (during matter domination). The field equation of motion (8) becomes

$$x' = -x + \frac{-5c_3\bar{H}^4x^2 + 30c_4\bar{H}^6x^3 - 75c_5\bar{H}^8x^4/2}{c_2\bar{H}^2/6 - 2c_3\bar{H}^4x + 9c_4\bar{H}^6x^2 - 10c_5\bar{H}^8x^3} \quad (27)$$

during radiation domination and

$$x' = -x + \frac{-c_2\bar{H}^2x/12 - 7c_3\bar{H}^4x^2/2 + 45c_4\bar{H}^6x^3/2 - 115c_5\bar{H}^8x^4/4}{c_2\bar{H}^2/6 - 2c_3\bar{H}^4x + 9c_4\bar{H}^6x^2 - 10c_5\bar{H}^8x^3} \quad (28)$$

during matter domination. We note the existence of a solution $x \sim \bar{H}^{-2}$ during both radiation and matter domination; this behaviour was first observed in [32]. However, solutions close to this fixed point at early times require an extraordinary fine tuning of initial conditions to ensure that the π field energy density remains subdominant until late times, as the energy density of the π field grows at a fast rate relative to matter and radiation (the π field equation of state is $w = -7/3$ and $w = -2$ during radiation and matter domination respectively.) We are interested in the behaviour of the π field for generic initial conditions.

Since each c_{2-5} contribution to the scalar field energy density enters in the ratio of \bar{H}^2x relative to the previous one, whether the initial condition has $x_i \ll \bar{H}_i^{-2}$ or not determines the early time evolution. For example, if this holds then the lowest, c_2 term will dominate ρ_π and $x \sim a^{-1}$, $x \sim a^{-3/2}$ during radiation and matter domination respectively. Since the other terms then decay at a faster rate than the c_2 term, once c_2 dominates it will do so at all subsequent times. In this case the π energy density will decay as a massless scalar field $\rho_\pi \sim a^{-6}$ at all times (recall c_2 involves the canonical kinetic term), leading to no interesting dynamics.

As we increase x_i the c_{3-5} terms will dominate. This is expected in the early Universe, where the model will be in the strongly (self)coupled regime. Note that the dynamics of ρ_π is hierarchical: for example if the c_3 term dominates at a particular time then the $c_{4,5}$ terms will remain subdominant at all subsequent times, however the c_2 term grows relative to the c_3 term and can eventually dominate.

During the radiation epoch, we find the following behaviour of x and ρ_π , depending on which of the c_{2-5} terms dominate:

$$c_5 : \quad x \sim a^{11/4} \quad \rho_\pi \sim c_5 a^{-9/4} \quad w = -1/4 \quad (29)$$

$$c_4 : \quad x \sim a^{7/3} \quad \rho_\pi \sim c_4 a^{-8/3} \quad w = -1/9 \quad (30)$$

$$c_3 : \quad x \sim a^{3/2} \quad \rho_\pi \sim c_3 a^{-7/2} \quad w = 1/6 \quad (31)$$

$$c_2 : \quad x \sim a^{-1} \quad \rho_\pi \sim c_2 a^{-6} \quad w = 1 \quad (32)$$

and during the matter epoch

$$c_5 : \quad x \sim a^{15/8} \quad \rho_\pi \sim c_5 a^{-21/8} \quad w = -1/8 \quad (33)$$

$$c_4 : \quad x \sim a^{3/2} \quad \rho_\pi \sim c_4 a^{-3} \quad w = 0 \quad (34)$$

$$c_3 : \quad x \sim a^{3/4} \quad \rho_\pi \sim c_3 a^{-15/4} \quad w = 1/4 \quad (35)$$

$$c_2 : \quad x \sim a^{-3/2} \quad \rho_\pi \sim c_2 a^{-6} \quad w = 1. \quad (36)$$

The effective equation of state of the π field varies between $w \in [-1/4, 1/4]$ when the c_{3-5} terms dominate during these epochs. During radiation domination, the π energy density typically grows relative to the matter component, however this effect is generally small and does not require a significant tuning of $\Omega_\pi(z_i)$ initially. For example, if the c_5 term provides the main contribution to ρ_π during radiation domination then we have $\rho_\pi/\rho_m \sim a^{3/8}$. This is the worst case scenario, and provides a limit on how large we can set ρ_π initially such that it remains subdominant at matter radiation equality (the exact constraint depends on our choice of z_i). Thus Galileon models possess the interesting property that they can belong to the class of early dark energy models, in the sense that their energy density is non-negligible at early times for generic initial conditions. (Indeed, the c_4 dominating case maintains a constant fraction of the matter density during matter domination.)

To calculate the behaviour of the π field at late times one can obtain the asymptotic de Sitter fixed points using a dynamical systems approach based on Eqs. (8)-(9). The fixed points correspond to solutions of the algebraic relations

$$\lambda = -\omega x \quad (37)$$

$$\gamma = -\beta x \quad (38)$$

that are coupled polynomials for \bar{H}^2 and x . For any values of the parameters c_{2-5} there is a double zero corresponding to Minkowski space; $\bar{H} = 0$. In addition, one can show that there are at most three de Sitter, three anti de Sitter and six unphysical complex solutions for \bar{H} . To see this, we can use the coupled parameter $\chi = \bar{H}^2 x$, in which case Eq. (38) reduces to the cubic polynomial

$$15c_5\chi^3 - 18c_4\chi^2 + 6c_3\chi - c_2 = 0, \quad (39)$$

which has three (one) real solutions if its discriminant

$$D = 144c_3^2c_4^2 - 160c_3^3c_5 + 360c_2c_3c_4c_5 - 288c_2c_4^3 - 75c_2^2c_5^2 \quad (40)$$

is positive (negative). One can also state that for $c_5 < 0$ (as needed for positive energy density when c_5 dominates early) there is exactly one real solution for x whenever $5c_3c_5 - 6c_4^2 > 0$. Note that this does not necessarily imply the existence of a de Sitter point, which can only be determined by solving Eq. (37). In terms of χ , the condition for the existence of a de Sitter vacuum state is

$$3\bar{H}^4 = -6c_5\chi^5 + \frac{9}{2}c_4\chi^4 - \frac{c_2}{2}\chi^2 > 0. \quad (41)$$

Hence for any real χ , there is at most one positive and one negative real \bar{H} solution to Eq. (41), and there are at most three distinct, real χ solutions to Eq. (39). The minimal number of de Sitter solutions is zero.

Certain specific cases are worth mentioning. We require at least two non-zero c_{2-5} terms to admit a de Sitter fixed point, and during the approach to the de Sitter asymptote, generically all terms are of the same order. Restricting our analysis to the case that any two of the terms c_{2-5} are non-zero, we can succinctly write the conditions for positivity of ρ_π at early times and the existence of a late time de Sitter point as: $(c_2, c_3) = (-, -)$, $(c_2, c_4) = (-, +)$, $(c_2, c_5) = (-, -)$. In each case, there is exactly one de Sitter point (although in the c_4 case the solution has multiplicity two). Note that the standard kinetic term must exist (or the no-ghost condition of Sec. V A is violated) and must take the non-canonical sign.

Both the early time and late time analytic results are borne out by the numerical solutions of the dynamical equations. Throughout this article we take initial conditions at $z_i = 10^6$ of $\rho_\pi(z_i) = 10^{-5}\rho_m(z_i)$ and fix $\Omega_m(z=0) = 0.24$. The parameters, and behaviors, exhibited in Figure 1 are typical in the sense elucidated in Section V D.

B. Linearly Coupled Galileon: $c_0 \neq 0, c_G = 0$

We now consider a linear coupling between the field π and the trace of the energy-momentum tensor (or Ricci scalar) [26, 42]. As mentioned, such a scenario is typical in higher dimensional braneworld models, and in particular

in the decoupling limit of the DGP model. We begin as before by examining the behaviour of the π field during matter and radiation domination. We initially consider the DGP like case where $c_{4,5} = 0$ and then extend to include the more general Galileon kinetic terms.

In the decoupling limit, the DGP model can be written as a four dimensional effective action with a scalar field coupled to the Ricci scalar and containing kinetic terms of the form $c_{2,3}$. With the parameters c_0, c_2, c_3 we expect to recover DGP-like behaviour in the early universe, where ρ_π is subdominant and we can neglect the backreaction of the π field on the geometry. Indeed, if we choose initial conditions such that $c_0 y \ll 1$, we find two approximate solutions during matter domination of the form $\bar{H} \sim a^{-3/2}$, $x = \pm A_0 \bar{H}^{-1}$, where $A_0 = \sqrt{c_0/(3c_3)}$. On this solution, $\rho_\pi \sim \pm 4c_0 A_0 \bar{H}$, the familiar modification to the Friedmann equation one would expect from a model mimicking the DGP model (here we choose the sign convention $c_0, c_3 > 0$).

On the “self accelerating branch” corresponding to $x > 0, \rho_\pi > 0$, one can also see that the π field does not satisfy the no-ghost condition given by (see Sec. V A)

$$(c_2 - 12c_3 \bar{H}^2 x) (1 - 2c_0 y) + 6 (c_3 \bar{H}^2 x^2 - c_0)^2 > 0. \quad (42)$$

At early times the first term on the left hand side dominates, and we can neglect the c_2 and $c_0 y$ contributions, leaving just $-12c_3 \bar{H}^2 x > 0$. For positive c_3 , the sign of x dictates whether the field is a ghost; on the self accelerating branch the inequality is not satisfied and π is a ghost as expected [54, 55].

We now introduce the $c_{4,5}$ terms, and consider how the evolution of π is modified. To exhibit the behaviour of x at early times, we write $\bar{H} \sim a^{-2}$ and $\bar{H} \sim a^{-3/2}$ for radiation and matter domination. But it should be observed that the standard behaviour of \bar{H} is no longer guaranteed: from the Friedmann equation (17) it is clear that to obtain the standard behaviours $\bar{H} \sim a^{-2}$ and $\bar{H} \sim a^{-3/2}$ we must impose $c_0 y \ll 1$ to avoid modification of the left hand side. We therefore set $y = 0$ initially, at some redshift deep in radiation domination, and check that y remains small.

During radiation domination, we find the same solutions (29)-(32) as in the uncoupled case due to the fact that the c_0 coupling term is subdominant in the x equation, so

$$x' = -x + \frac{-5c_3 \bar{H}^4 x^2 + 30c_4 \bar{H}^6 x^3 - 75c_5 \bar{H}^8 x^4/2}{c_2 \bar{H}^2/6 - 2c_3 \bar{H}^4 x + 9c_4 \bar{H}^6 x^2 - 10c_5 \bar{H}^8 x^3}. \quad (43)$$

However during matter domination we find modified behaviour,

$$x' = -x + \frac{-c_2 \bar{H}^2 x/12 - 7c_3 \bar{H}^4 x^2/2 + 45c_4 \bar{H}^6 x^3/2 - c_0 \bar{H}^2/2 - 115c_5 \bar{H}^8 x^4/4}{c_2 \bar{H}^2/6 - 2c_3 \bar{H}^4 x + 9c_4 \bar{H}^6 x^2 - 10c_5 \bar{H}^8 x^3}. \quad (44)$$

The behaviour of x and hence ρ_π depends upon the initial conditions imposed, and general analytic solutions do not exist. However for certain limiting cases one can construct explicit solutions. One example is when the c_3 term dominates, corresponding to the DGP like solution discussed above with $\rho_\pi/(H_0^2 M_{\text{pl}}^2) = 4c_0 A_0 \bar{H}$. Analogous solutions exist for the cases where other terms dominate; note that again the contributions to ρ_π are hierarchical – if the c_n term dominates early, it can only be superceded later in matter domination by a c_m term with $0 < m < n$. The matter dominated era solutions, depending on which of the c_{2-5} terms dominate alongside c_0 , are

$$c_5, c_0 : \quad x = A_0 \bar{H}^{-3/2} \quad \rho_\pi = \frac{16}{5} c_0 A_0 \bar{H}^{1/2} \quad A_0 = \left(\frac{2c_0}{15c_5} \right)^{1/4} \quad w = -\frac{11}{8} \quad (45)$$

$$c_4, c_0 : \quad x = A_0 \bar{H}^{-4/3} \quad \rho_\pi = \frac{7}{2} c_0 A_0 \bar{H}^{2/3} \quad A_0 = \left(-\frac{c_0}{9c_4} \right)^{1/3} \quad w = -\frac{26}{21} \quad (46)$$

$$c_3, c_0 : \quad x = A_0 \bar{H}^{-1} \quad \rho_\pi = 4c_0 A_0 \bar{H} \quad A_0 = \left(\frac{c_0}{3c_3} \right)^{1/2} \quad w = -1 \quad (47)$$

$$c_2, c_0 : \quad x = A_0 \quad \rho_\pi = 5c_0 A_0 \bar{H}^2 \quad A_0 = \frac{-2c_0}{c_2} \quad w = -\frac{2}{5}, \quad (48)$$

where we have chosen sign conventions such that $x > 0$. Note that because of the explicit coupling to the Ricci scalar, the standard continuity equation and $\rho_\pi \sim a^{-3(1+w)}$ do not hold. We also note that the last case is only viable if $|c_0^2/c_2| \ll 1$, otherwise we will pick up non-negligible corrections to the standard matter era due to the presence of the $c_0 y$ term on the left hand side of the Friedmann equation. We will show in Appendix D that in each of the other cases, the π field is a ghost whenever $\rho_\pi > 0$ (this result does not depend on our choice $x > 0$). Therefore the linear coupled model is only viable when either the c_2 term dominates the $c_{m>2}$ terms at early times and $|c_0^2/c_2| \ll 1$, or when the c_{2-5} contributions to the π energy density are much larger than that from c_0 . In the latter scenario the model will evolve approximately as in the uncoupled case during matter and radiation domination.

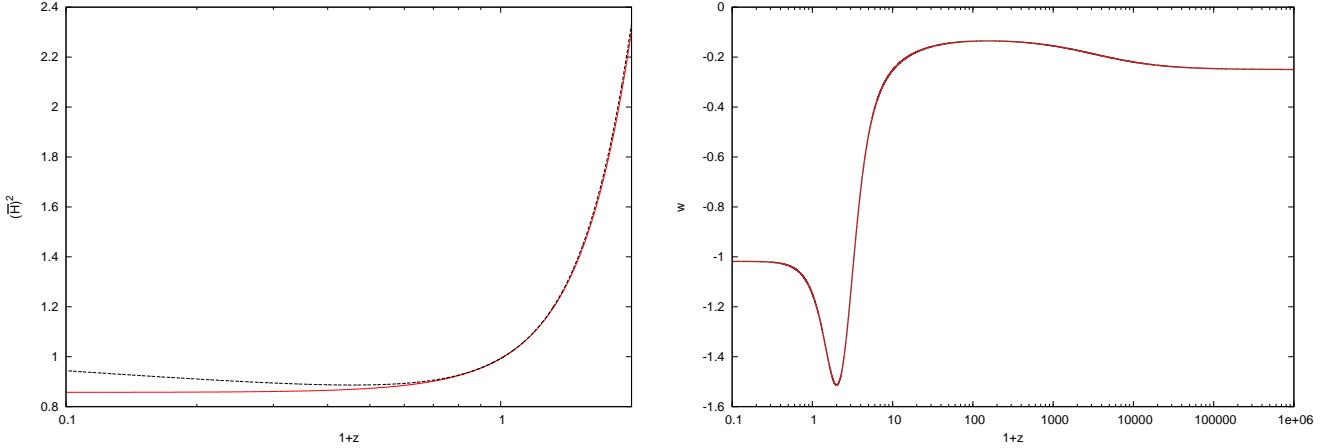


FIG. 2: [Left panel] The redshift evolution of viable linear coupled Galileons is restricted to be near that of the equivalent uncoupled Galileon, except at late times. The late time behaviour of \bar{H}^2 for the linearly coupled model (dotted black) is compared to the uncoupled model \bar{H}^2 (solid red) with the parameter choices $(c_2, c_3, c_4, c_5, c_0) = (-5.9, -2, -1, -1, 0.05)$ and $(-6.0, -2, -1, -1, 0)$ respectively (c_2 is adjusted to ensure $\Omega_{\pi,0} = 0.76$ in both cases). Note the Hubble parameter in the linear coupling case no longer asymptotes to a constant value. [Right panel] The effective equation of state for the linearly coupled π field as a function of redshift. At early times w resembles the uncoupled case, e.g. Fig. 1, but deviates at later times and in the future has w close to but not exactly -1 .

At late times we no longer obtain a de Sitter fixed point; this is due to the fact that the field y now enters the Friedmann equation explicitly. The de Sitter solution in the uncoupled case corresponds to \bar{H}, x approaching constant values asymptotically, and this is no longer possible since then $c_0 y$ would grow. However, since y grows as $\ln a$ for constant x , a near de Sitter state is possible, where $w \sim -1$ for $z \lesssim -0.5$. From our numerical studies we find that typical deviations from $w = -1$ are less than $\sim 5\%$ for model parameters in the range $c_{2-5} \sim (-10, 10)$ and $c_0 \sim (-1, 1)$ at $z \sim -0.95$ (though larger deviations are possible).

The numerical evolution of the linearly coupled model is exhibited in Figure 2 for generic parameter values $c_0 = 0.1$, $c_3 = -1$, $c_4 = 1$, $c_5 = -1$, which fixes $c_2 = -10.0$ so that $\Omega_\pi(z=0) = 0.76$. The high redshift evolution is necessarily similar to the uncoupled case, as just discussed, but the future evolution differs. We note that \bar{H} and x continue to evolve in the future and do not approach constant values as in the uncoupled case, so there is no de Sitter state; nevertheless the equation of state w remains very close to -1 .

C. Derivative Coupled Galileon: $c_G \neq 0, c_0 = 0$

In this subsection we include the derivative coupling term with c_G , and switch off the linear coupling term with c_0 . Recall that the kinetic coupling to the Einstein tensor, $G^{\mu\nu}\partial_\mu\pi\partial_\nu\pi$, also arises in some higher dimension gravity theories and disformal field theories and by itself has interesting behavior involving cosmic acceleration [44, 45].

At early times, the c_G contribution will dominate over the c_2 one, and for reasonable initial conditions (i.e. that the energy density $\rho_\pi \ll \rho_m$ initially) x will satisfy $x \ll 1$ and hence the c_G term will dominate over the c_3 term as well. However the importance of the c_G term relative to $c_{4,5}$ depends on the specific initial conditions. If c_G does not dominate, then we have the previous uncoupled case behaviors. If c_G does dominate then during radiation domination

$$c_G : \quad x \sim a^3 \quad \rho_\pi \sim c_G a^{-2} \quad w = -1/3 \quad (49)$$

and during matter domination

$$c_G : \quad x \sim a^{3/2} \quad \rho_\pi \sim c_G a^{-3} \quad w = 0. \quad (50)$$

Note that ρ_π grows relative to the matter energy density during the radiation epoch and scales with it for $z \lesssim 10^3$, and therefore we must be careful to choose initial conditions such that ρ_π remains subdominant until $z \sim 1$. In particular, we impose that $\Omega_\pi \lesssim 2 \times 10^{-2} \Omega_m$ at matter/radiation equality. We exhibit the full behaviour in Figure 3, numerically evolving the equations for the case $c_4 = c_5 = 0$, $c_3 = -1$ and $c_G = -1$. The transition from $w = -1/3$ during radiation domination to $w = 0$ for $z < 1000$ is clear. If we switch on the $c_{4,5}$ terms then since those terms grow relative to the c_G term during radiation domination the limiting cases of Eqs. (49) and (50) will no longer be clearly distinguishable.

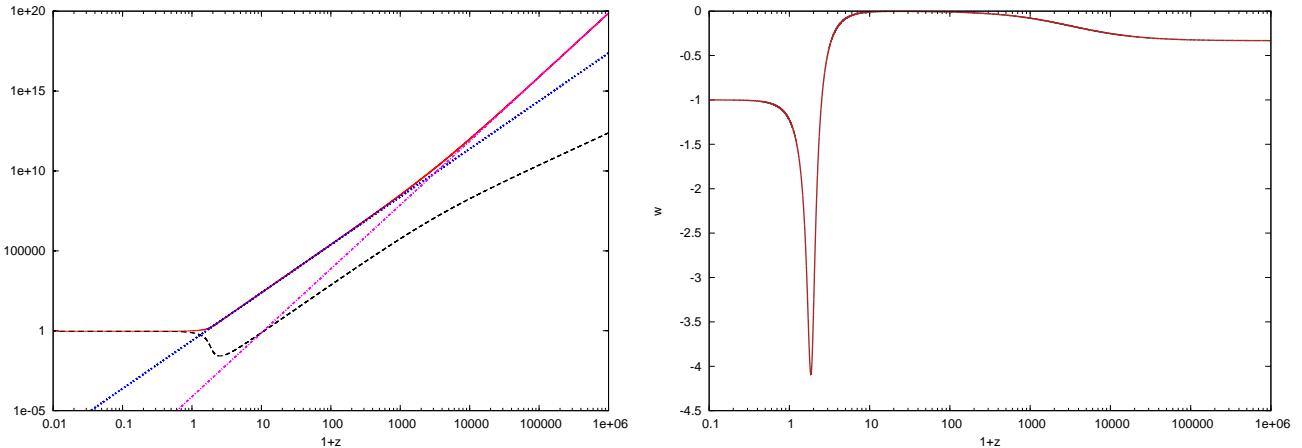


FIG. 3: [Left panel] The evolution of \bar{H}^2 (solid red), $8\pi G\rho_m/3H_0^2$ (dotted blue), $8\pi G\rho_r/3H_0^2$ (dot-dashed magenta) and $8\pi G\rho_\pi/3H_0^2$ (long dash black) for the derivatively coupled Galileon, with $c_2 = -9.1$, $c_3 = -1$, $c_4 = c_5 = 0$. [Right panel] The equation of state of the π field as a function of redshift. The numerical evolution exhibits the analytic attractor results.

At late times the derivative coupled Galileon goes to the de Sitter attractor. Note that in the model of [44] using derivative coupling, the c_2 term was forced to the canonical form, and so no stable de Sitter attractor was allowed, however here there is sufficient freedom to enable an asymptotic de Sitter state.

V. EVOLUTION OF PERTURBATIONS AND GRAVITATIONAL STRENGTH

Having established the behaviour of the homogeneous cosmological expansion for the classes of Galileon models, we now examine the behaviour of linearized perturbations. Since the derivatives of the π field couple to the metric potentials, we expect the field to modify the growth of matter density perturbations by introducing time dependent couplings between $\delta\pi$, ϕ and ψ . The equations governing the behaviour of subhorizon perturbations are given in Appendix C. As discussed in Section III the effects on growth of structure (and light deflection) are concisely encapsulated in the gravitational strength modification functions $G_{\text{eff}}^{(\psi)}$ and $G_{\text{eff}}^{(\psi+\phi)}$. It has been shown in [40, 41] that stringent constraints can be placed on the Galileon class of models by considering the time evolution of G_{N} .

The “paths of gravity” describe the evolution of these functions, from the initial deviation from the general relativistic value of G_{N} at high redshift to present day signatures and the eventual asymptotic values. It is important to stress the domain of validity of our equations; they are applicable to linear perturbations on subhorizon scales, assuming that dark matter and baryons can be treated as a common fluid. We consider the evolution of perturbations over the redshift range $z < 500$. We set initial conditions at $z = 10^6$ as $\rho_\pi = 10^{-5}\rho_m$, to ensure that at $z \sim 500$ the perturbations remain small and G_{eff} is sufficiently close to its general relativistic behaviour. At low redshifts, linear perturbation theory is only applicable for large scale modes. To model non-linear perturbations at the present time we would be forced to resort to simulations, which is beyond the scope of the present work. Significant deviations from the linearized behaviour will be expected in the non-linear regime, due to the presence of a cosmological Vainshtein screening effect.

The perturbation equations for the π field itself must also satisfy certain physical conditions for a sound theory, such as the absence of ghosts modes and Laplace instabilities. These will constrain the allowed parameter space, as can conditions placed on the sign and magnitude of G_{eff} . Before discussing the gravitational evolution we briefly discuss the no-ghost condition, later treating its details and the Laplace instability in Appendices D and E.

A. No-Ghost Condition

The equation of motion for the field perturbations takes the form of a wave equation, with the sign of the second time derivative $\delta\ddot{y}$ needing to be non-negative to avoid ghosts (negative kinetic terms causing the Hamiltonian to be unbounded from below). For the standard, simple case of a minimally coupled, canonical scalar field the condition is just positivity of the kinetic energy, i.e. $c_2 > 0$ when all other c 's are zero. In the presence of nonlinear kinetic terms in the action, and couplings, the situation is more complicated and we present our analysis in Appendix D. Here we

summarize the result: to be free of ghosts the theory must satisfy

$$-\frac{c_2}{2} + 6c_3\bar{H}^2x + 3c_G\bar{H}^2 - 27c_4\bar{H}^4x^2 + 30c_5\bar{H}^6x^3 + 2\frac{(3c_3\bar{H}^2x^2 + 6c_G\bar{H}^2x - 18c_4\bar{H}^4x^3 + \frac{45}{2}c_5\bar{H}^6x^4 - 3c_0)^2}{-6(1 - 2c_0y) - 6c_G\bar{H}^2x^2 + 9c_4\bar{H}^4x^4 - 18c_5\bar{H}^6x^5} < 0. \quad (51)$$

B. The Thawing of Gravity at Early Times

At early times, with the Galileon contributions to the energy density small, the gravitational strength functions go to the general relativistic values of unity. That is, the early universe behaves as in standard gravity and cosmology. As the Galileon energy density increases, the modifications to gravity grow; we can say that the theory thaws away from general relativity, in analogy to the thawing class of dark energy dynamics (where the scalar field moves away from a frozen, cosmological constant state).

For the initial stages of thawing we can calculate both G_{eff} and the no-ghost condition analytically for each of the cases where a given term of $c_{2-5,G}$ dominates the π energy density. For the no-ghost condition, we find

$$c_5 : -\frac{\bar{\rho}_\pi}{\bar{H}^2} - \frac{15}{56} \left(\frac{\bar{\rho}_\pi}{\bar{H}^2} \right)^2 < 0 \quad (52)$$

$$c_4 : -\frac{\bar{\rho}_\pi}{\bar{H}^2} - \frac{8}{45} \left(\frac{\bar{\rho}_\pi}{\bar{H}^2} \right)^2 < 0 \quad (53)$$

$$c_3 : -\frac{\bar{\rho}_\pi}{\bar{H}^2} - \frac{1}{12} \left(\frac{\bar{\rho}_\pi}{\bar{H}^2} \right)^2 < 0 \quad (54)$$

$$c_2 : -\frac{\bar{\rho}_\pi}{\bar{H}^2} < 0 \quad (55)$$

$$c_G : -\frac{\bar{\rho}_\pi}{\bar{H}^2} - \frac{4}{9} \left(\frac{\bar{\rho}_\pi}{\bar{H}^2} \right)^2 < 0 \quad (56)$$

where $\bar{\rho}_\pi = \rho_\pi/(H_0^2 M_{\text{pl}}^2)$ is the dimensionless Galileon energy density. In all cases, the condition $\rho_\pi > 0$ is sufficient to ensure that the field is not a ghost. Note that given the initial conditions such that $\bar{\rho}_\pi \ll \bar{H}^2$ during matter and radiation domination, the first terms in each will always dominate.

For the gravitational strength, we find

$$c_5 : \frac{G_{\text{eff}}^{(\phi)}}{G_N} = 1 + \frac{48}{7}\Omega_\pi \quad (57)$$

$$c_4 : \frac{G_{\text{eff}}^{(\phi)}}{G_N} = 1 + \Omega_\pi \quad (58)$$

$$c_3 : \frac{G_{\text{eff}}^{(\phi)}}{G_N} = 1 + \frac{1}{5}\Omega_\pi \quad (59)$$

$$c_2 : \frac{G_{\text{eff}}^{(\phi)}}{G_N} = 1 \quad (60)$$

$$c_G : \frac{G_{\text{eff}}^{(\phi)}}{G_N} = 1 + \Omega_\pi. \quad (61)$$

These analytic solutions are only valid during matter domination. Figure 4 exhibits the evolution of the deviations from general relativity $G_{\text{eff}}/G_N - 1$ for each of the effective gravitational strengths, from an early time where they share similar redshift evolution in the deviation from general relativity, to a shared late time de Sitter asymptote as discussed in the following section.

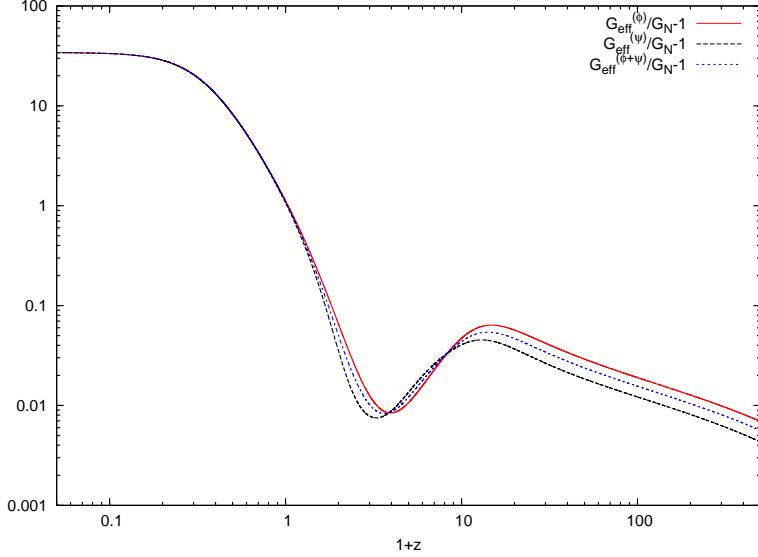


FIG. 4: The thawing of gravity away from its general relativistic strength, $G_{\text{eff}}^{(\phi)}/G_N - 1$ (solid red), $G_{\text{eff}}^{(\psi)}/G_N - 1$ (long dash black), and $G_{\text{eff}}^{(\phi+\psi)}/G_N - 1$ (short dash blue) for model parameters $(c_2, c_3, c_4, c_5) = (-15.3, -5.73, -1.2, -1)$. All three functions share a common redshift dependence at early times and a common future asymptote, as discussed in the text.

C. Gravity in the de Sitter Limit

The other limit of importance is the late time approach to the de Sitter attractor that is present in the uncoupled and derivatively coupled models. In the de Sitter limit, we have $\bar{H}', x' \rightarrow 0$, and it is convenient to define

$$C \equiv c_2 x^2 \quad (62)$$

$$D \equiv c_3 \bar{H}^2 x^3 \quad (63)$$

$$E \equiv c_4 \bar{H}^4 x^4 \quad (64)$$

$$F \equiv c_5 \bar{H}^6 x^5 \quad (65)$$

$$A \equiv c_G \bar{H}^2 x^2. \quad (66)$$

Note that generically all these terms are of the same order (unless a coefficient is zero) so we cannot study the limit in terms of a single dominant component, making the analysis more difficult but the behavior richer.

In general (with one type of coupling) we have five parameters and two independent equations of motion. The background field equations then reduce to algebraic constraints:

$$E = \frac{1}{9} (-2A + 8D - 3C - 10) \quad (67)$$

$$F = \frac{2}{3} \left(D - A - \frac{C}{2} - 2 \right). \quad (68)$$

Eliminating E, F from $G_{\text{eff}}^{(\phi)}$, we arrive at the de Sitter limit

$$G_{\text{eff}, \text{dS}} = -\frac{3}{14 + 7A + 3C - 4D} G_N. \quad (69)$$

Interestingly, in the de Sitter limit $G_{\text{eff}}^{(\phi)} = G_{\text{eff}}^{(\psi)} = G_{\text{eff}}^{(\phi+\psi)}$ – a special property of the Galileon models that in this limit they have no gravitational slip – and we write them simply as $G_{\text{eff}, \text{dS}}$.

It is also useful to write the no-ghost condition at the de Sitter state (although the no-ghost condition must hold at all times) in terms of A, C, D . We find

$$-\frac{3}{2}C + 2D - 11A - 10 + 2 \frac{(-\frac{3}{2}C + 2D - 5A - 10)^2}{8 + 3C - 4D + 4A} < 0. \quad (70)$$

For the uncoupled case $A = 0$, the no-ghost condition corresponds to

$$2 + \frac{3C}{4} < D < 5 + \frac{3C}{4}. \quad (71)$$

Figure 5 illustrates the constraints imposed on the Galileon parameter space by several conditions for physical viability, including the no-ghost condition, Laplace stability condition on the sound speed, and positivity of energy density ρ_π , and their relation to the asymptotic de Sitter value of the gravitational strength $G_{\text{eff,ds}}$. For the uncoupled case note that $c_5 < 0$, which is generally required in the early universe barring fine tuning, requires that both c_3 and c_4 must also be negative.

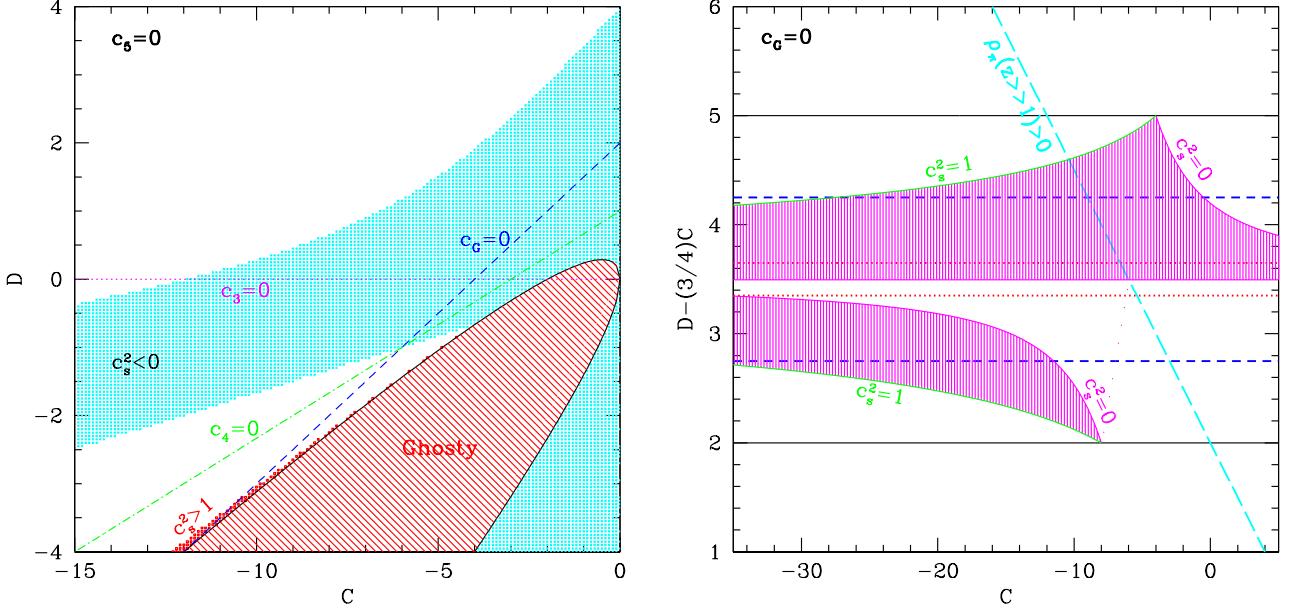


FIG. 5: [Left panel] Dark red shaded region shows the area violating the no-ghost condition in the C - D plane, fixing $c_5 = 0$. The straight lines show the cuts through the plane corresponding to special values $c_n = 0$ as labeled. The light cyan shaded region shows the Laplace instability $c_s^2 < 0$, and the thin “bubbly” red region shows the superluminal condition $c_s^2 > 1$ (see Appendix E). $C > 0$ is also ruled out. [Right panel] For the $c_6 = 0$ case the no-ghost region lies between the solid black lines. These boundaries correspond to the limits $G_{\text{eff,ds}}/G_N = \pm 1/2$; the blue dashed lines give $G_{\text{eff,ds}}/G_N = \pm 1$, and the red dotted lines give $G_{\text{eff,ds}}/G_N = \pm 5$. Positive values are for the upper line of each set. $G_{\text{eff,ds}}$ passes through infinity for $4D - 3C = 14$. Note $G_{\text{eff,ds}}/G_N$ cannot lie between $-1/2$ and $1/2$. Further constraints include positivity of scalar field energy density at high redshift, limiting the parameter space to the left of the long dashed, diagonal cyan line when the c_5 term dominates, and the Laplace inequality (E3), with shaded magenta regions obeying $0 \leq c_s^2 \leq 1$.

Since after the equations of motion are applied one has G_{eff} as a function of three free parameters, it is difficult to make general analytic statements. We begin by considering restricted cases, and then carry out a more general scan through the full parameter space.

First suppose that only one of the parameters $c_{2-5,G}$ is nonzero. We find that in no case is there a consistent de Sitter solution. Solutions are possible when, e.g., two or three $c_{2-5,G}$ terms are nonzero. For a more physically motivated restriction, one can look at parameter choices for which gravity returns asymptotically to general relativity, at least in the sense of $G_{\text{eff,ds}} = 1$. This requires the condition

$$17 + 7A + 3C - 4D = 0, \quad (72)$$

so now our 5 dimensional parameter space is restricted to 2 dimensions and we can plot allowed regions, imposing as well the viability conditions. The no-ghost condition becomes

$$3C - 4D < \frac{-39 - 7\sqrt{57}}{6} \quad \text{or} \quad 4 > 3C - 4D > \frac{-39 + 7\sqrt{57}}{6}, \quad (73)$$

(although the second region violates Laplace stability, having $c_s^2 < 0$) and is exhibited in the left panel of Figure 6. Setting some $c_n = 0$ collapses the 2 dimensional space to a 1 dimensional line.

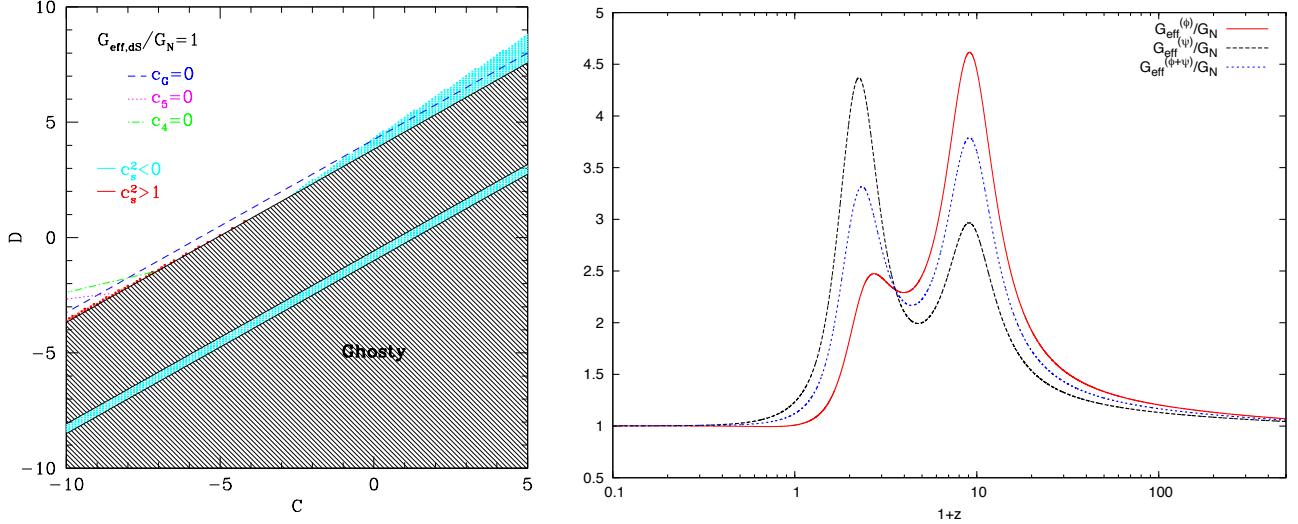


FIG. 6: [Left panel] Region of the Galileon parameter space that gives a de Sitter limit of $G_{\text{eff}}/G_N = 1$ and has no ghosts or Laplace instability is in the unshaded region. The 2 dimensional viable region becomes a 1 dimensional line if further restrictions are imposed. Note the uncoupled case $c_G = 0$ is here always ghost free, though it is Laplace unstable, $c_s^2 < 0$, for insufficiently negative C . [Right panel] The evolution of the effective gravitational strengths $G_{\text{eff}}^{(\phi)}$ (solid red), $G_{\text{eff}}^{(\psi)}$ (long dash black) and $G_{\text{eff}}^{(\phi+\psi)}$ (short dash blue) for an uncoupled Galileon case restoring to general relativity in the de Sitter limit, with and $(c_2, c_3, c_4, c_5) = (-12.688, -6.239, -2.159, -1)$. Note the strengths can be significantly different from each other.

D. The Paths of Gravity

Having constructed some limiting cases analytically, we now examine the full parameter space by scanning through it, solving for G_{eff} numerically. Our aim is to find generic trends in the behaviour of $G_{\text{eff}}^{(\phi)}$ and $G_{\text{eff}}^{(\psi)}$, and furthermore restrict the parameter space by applying theoretical constraints on the behaviour of the π field.

As an initial illustration of the paths of gravity, i.e. the evolution of the gravitational strength $G_{\text{eff}}(a)$, we consider the family of models that result in $G_{\text{eff,ds}}/G_N = 1$. For the uncoupled case $c_G = c_0 = 0$ our approach is to fix $c_5 = -1$, and c_2 by demanding $\Omega_{\pi,0} = 0.76$. There remains a set of (c_3, c_4) values that can satisfy our remaining condition $G_{\text{eff,ds}}/G_N = 1$.

Figure 6, right panel, plots the effective gravitational strengths $G_{\text{eff}}^{(\phi)}(a)/G_N$, $G_{\text{eff}}^{(\psi)}(a)/G_N$, and $G_{\text{eff}}^{(\phi+\psi)}(a)/G_N$ for one case giving rise to the asymptotic behaviour $G_{\text{eff,ds}}/G_N = 1$. Despite having past and future asymptotes agreeing with general relativity, the intermediate evolution over the redshift range $z \approx (0, 10)$ can be strongly distinct, both from general relativity and among the different G_{eff} 's. The magnitude of the features over this range is extremely sensitive to the values of $c_{3,4}$.

While it is instructive to analyze particular examples of model parameters, we can also learn a lot by a broader if shallower study. Generically we have a three (uncoupled case) or four (linearly or derivatively coupled cases) dimensional parameter space spanned by c_{3-5} , plus possibly c_0 or c_G . As before we fix c_2 by requiring the present dark energy density $\Omega_{\pi,0} = 0.76$. In what follows we also fix $c_5 = -1$ for tractability, and hence as mentioned in the previous subsection only have to consider the range $c_3, c_4 < 0$. We systematically scan the parameter space by selecting 22,500 points from a uniform grid over the range $(-5, 0)$ for each of the $c_{3,4}$ terms. We discuss the c_0 and c_G parameters below. For each point we evolve the background equations (8, 9) from $z = 10^6$ to $z = -0.99$ (i.e. future scale factor $a = 100$). From the 22,500 expansion histories, we keep as viable only those that satisfy the following conditions for $z < 500$:

- (a) The energy density of the π field is positive definite, $\rho_\pi > 0$.
- (b) The no-ghost condition (D8) is satisfied.
- (c) The Laplace inequality (E2) is satisfied: $c_s^2 \geq 0$.

The positivity of the energy density is not strictly required at all times during the cosmological history. However, there are a number of reasons as to why we include it. We have already seen in (52 – 56) that at early times the

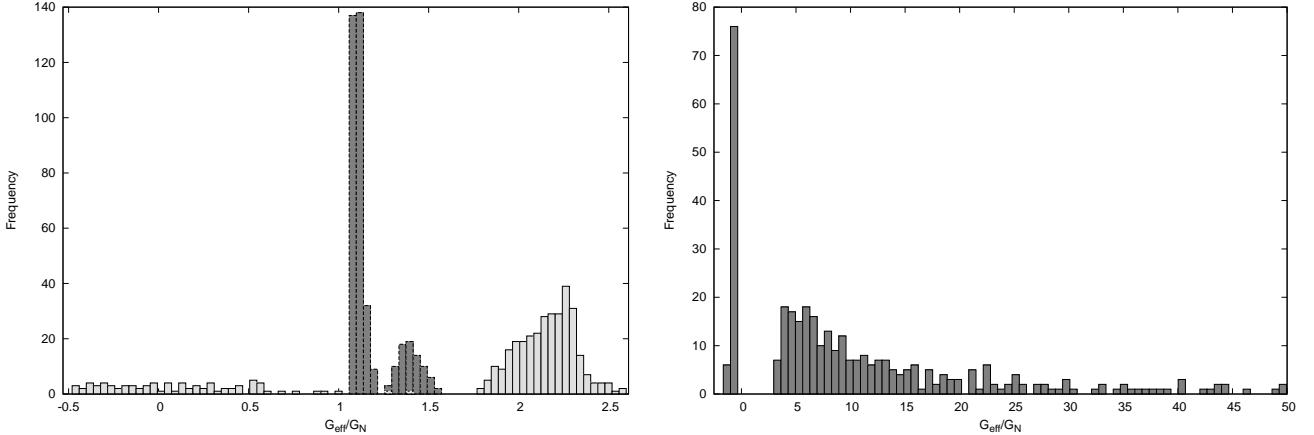


FIG. 7: [Left panel] Histogram of the gravitational strength $G_{\text{eff}}^{(\phi)}/G_N$ for an ensemble of theoretically viable models arising from a uniform scan through the uncoupled Galileon parameter space, at redshift $z = 1$ (dark) and $z = 0$ (light). At $z = 1$ we find G_{eff} to be clustered in the vicinity of its general relativistic limit (deviating by $\sim 10\%$), however at late times the behaviour is more diverse and further from the GR limit. We observe that G_{eff} can be both positive and negative at late times, although the majority of the parameter space surveyed approached $G_{\text{eff}} > 0$. [Right panel] The asymptotic value of G_{eff} at $z = -0.99$ (essentially the de Sitter asymptote), showing extremely diffuse behaviour in the future.

no-ghost constraint is intimately tied to the $\rho_\pi > 0$ condition. Similarly, we must have $\rho_\pi > 0$ at the present time since we demand that the field drives the current late time acceleration. This does not preclude the possibility that the energy density can be positive at high redshift, cross through $\rho_\pi = 0$ at some intermediate time and then return to a positive value at $z = 0$. It is very difficult to prove that such a crossing can never occur, however we have reason to believe that such behavior is highly unlikely. There are multiple fixed points that the galileon field can approach asymptotically; in addition to the de Sitter point there are both anti de Sitter and Minkowski vacuum states, and it is likely that any crossing of the zero line $\rho_\pi = 0$ will result in the galileon failing to approach the de Sitter state that we require. Hence we impose $\rho_\pi > 0$.

Figure 7 plots the histograms of the results for the gravitational strengths $G_{\text{eff}}^{(\phi)}$ at redshifts $z = 0$, $z = 1$, and $z = -0.99$ for each of the surviving, theoretically viable runs. At $z = 1$ the viable models are strongly clustered in the vicinity of $G_{\text{eff}}^{(\phi)}/G_N \gtrsim 1$, with deviations from the high redshift, general relativistic (GR) value by only $\sim 10\%$. However by $z = 0$, when cosmic acceleration is strong, gravity generically is quite distinct from GR and a diversity of behaviors is possible. At late times, essentially at the de Sitter state, the value of the gravitational strength is diverse, and takes both positive and negative values of $G_{\text{eff}}^{(\phi)}/G_N$ (cf. the right panel of Figure 5). If we were to fix $G_{\text{eff,ds}}/G_N = 1$, say, then we find much greater diversity at $z \approx 1$ but of course uniformity in the future.

We next perform an analogous numerical study for the linearly coupled Galileon model, expanding our parameter grid. We continue to fix $c_5 = -1$, but now consider 250,000 expansion histories over the range $c_3 = (-5, 5)$, $c_4 = (-5, 5)$ and $c_0 = (-5, 5)$. The results are exhibited in Figure 8. In this case, we no longer demand that our theoretical constraints are satisfied at all times, but rather only for $z > 0$. We do not consider $z < 0$ as there is no de Sitter fixed point for this model, so we would have to check infinitely far to the future, and $z = 0$ is a logical cutoff in choosing a domain of applicability.

The distribution of $G_{\text{eff}}^{(\phi)}$ is similar to the uncoupled case – recall from Sec. IV B that generally the c_0 contribution has to be subdominant at early times. However the later time behaviour is more diffuse. This is due to the fact that we now permit G_{eff} to diverge for $z < 0$. Such divergences can occur for any $z < 0$, and the value of G_{eff} at $z = 0$ is particularly sensitive to the existence and redshift of these features.

Finally, we repeat our analysis for the derivatively coupled Galileon model, expanding our parameter grid to include $c_G \neq 0$. We evenly distribute 250,000 points amongst the three dimensional space $c_3 = (-5, 5)$, $c_4 = (-5, 5)$, $c_G = (-5, 5)$. The resulting histograms of theoretically viable models are exhibited in Figure 9. We observe qualitatively similar behaviour to the uncoupled model at both $z = 1$ and $z = 0$, although we find considerably more diffuse behaviour at both $z = 0$ and $z = 1$, with a higher propensity towards negative asymptotes.

Over the course of our numerical study, we considered the range $c_G = (-5, 5)$. However we found no models which were deemed viable for $c_G < 0$. The theoretical constraint that causes the derivatively coupled model to fail in this regime is the Laplace condition of Appendix E, $c_s^2 > 0$. If we evolve our equations back to radiation domination in the early universe, whenever the c_G term is dominant then $c_s^2 = -1/3 + \mathcal{O}(\Omega_\pi(z))$, indicating a classical instability [56].

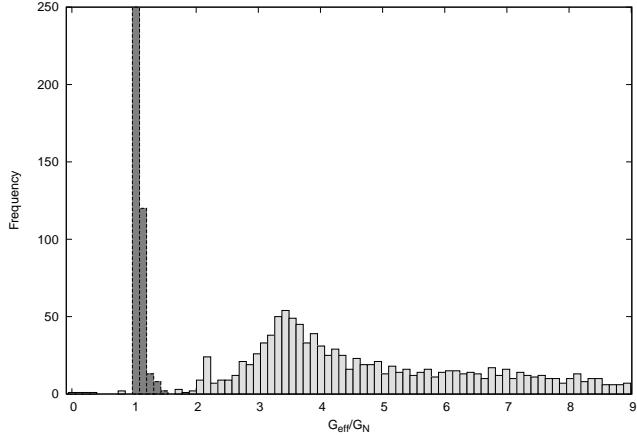


FIG. 8: Histogram of the gravitational strength $G_{\text{eff}}^{(\phi)}/G_N$ for an ensemble of theoretically viable models arising from a uniform scan through the linearly coupled Galileon parameter space, at redshift $z = 1$ (dark) and $z = 0$ (light). We do not here consider models that diverge in the future to be unviable, so greater diversity is exhibited compared to the uncoupled model.

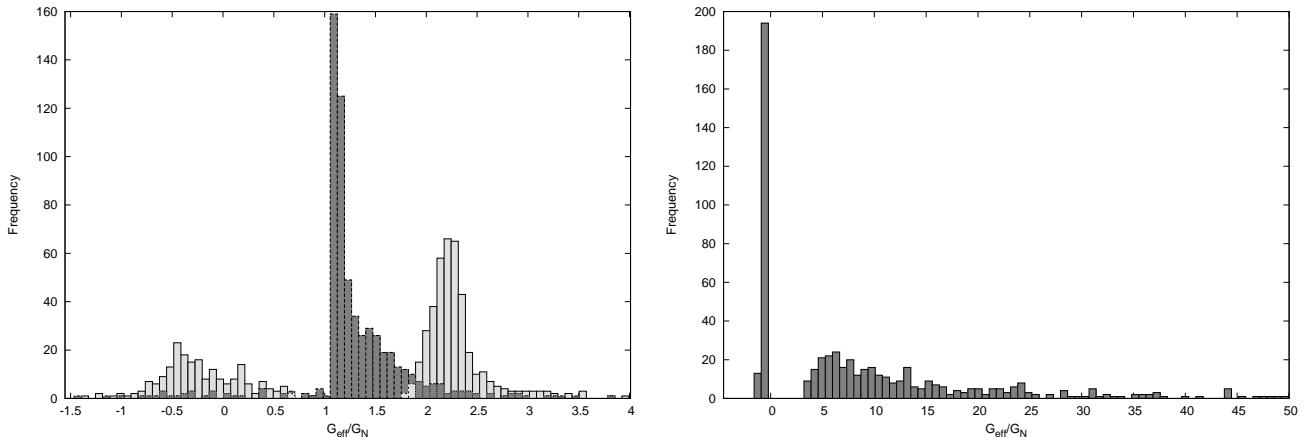


FIG. 9: Histogram of the gravitational strength $G_{\text{eff}}^{(\phi)}/G_N$ for an ensemble of theoretically viable models arising from a uniform scan through the derivatively coupled Galileon parameter space, at redshift $z = 1$ (dark) and $z = 0$ (light) in the left panel, and $z = -0.99$ in the right panel. We observe similar, although more diffuse, behaviour in the model with $c_G \neq 0$ compared to the uncoupled case.

Although we do not consider cosmological perturbations at such high redshifts in this work, the Laplace condition is applicable in the early universe since to leading order the metric perturbations decouple then, and we can simply consider perturbations of the Galileon field evolving on the background. In this case, $c_s^2 = -1/3$, but even during matter domination, whenever $\rho_{\text{rad}} > \rho_\pi$ we expect, and observe, this instability to be present, leading to the absence of viable models in our scans that differ substantially from the uncoupled case. An important consequence is that since the c_G term is always of order \bar{H}^2 larger than the c_2 term, a model with only c_2 and c_G , such as that of [44] (even generalized to arbitrary c_2 so as to enable a de Sitter future state), is ruled out. Also, the fact that only $c_G > 0$ models survive our constraints implies that this term cannot dominate ρ_π during matter and radiation domination.

In Figure 10 we exhibit the evolution of $G_{\text{eff}}^{(\phi)}(a)$ for a “typical” viable model – that is, a model near the peak of the histograms for $z = 1$ – for each of the uncoupled, linearly coupled, and derivatively coupled cases. We note a common deviation from general relativity at $z \approx 10$, a reapproach to GR, and then the strong departure as cosmic acceleration occurs. Evolving to the future beyond $z = 0$, $G_{\text{eff}}^{(\phi)}$ approaches a positive, constant value in the de Sitter asymptote for the derivative and uncoupled cases, while in the linear case $G_{\text{eff}}^{(\phi)}$ will continue to grow due to the explicit dependence on y .

The nonmonotonic behaviour of $G_{\text{eff}}^{(\phi)}$ at $z \approx 10$ is a common feature over the parameter space considered, and its origin can be traced to an interaction between the different c_n terms. Although the c_5 term, say, dominates at

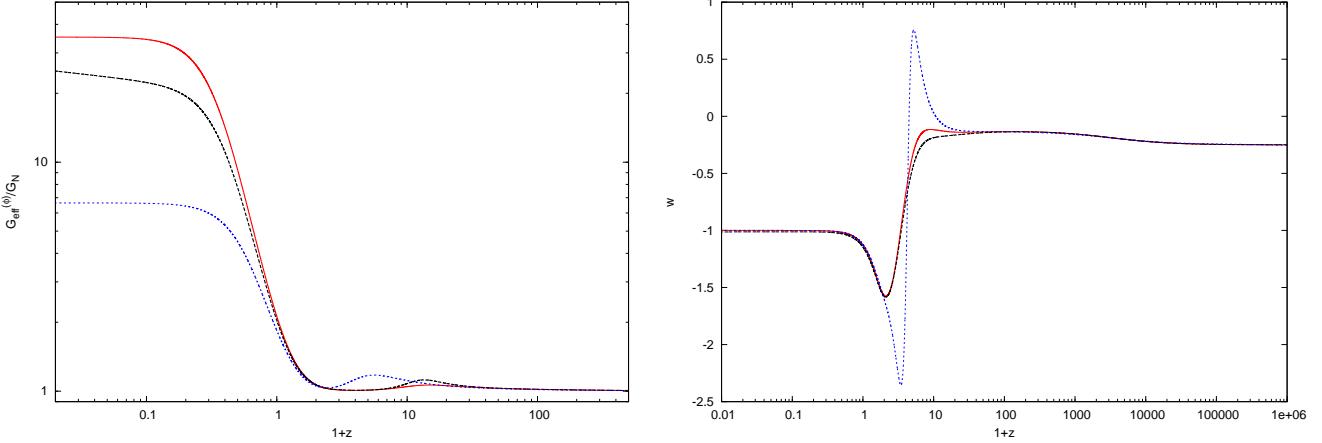


FIG. 10: Typical examples of [Left panel] $G_{\text{eff}}^{(\phi)}(a)$ and [Right panel] $w(a)$ for viable models in the uncoupled (solid red), linearly coupled (long dash black), and derivatively coupled (short dash blue) classes. For each of the cases we fix $c_3 = -5.73$, $c_4 = -1.2$ and $c_5 = -1$, and adjust the parameter c_0 or c_G such that the model exists in the vicinity of the peak of that case's $G_{\text{eff}}(z = 1)$ histogram (specifically $c_0 = 0.05$ or $c_G = 0.4$). The behaviour of uncoupled and derivatively coupled models is qualitatively similar, with both $G_{\text{eff}}^{(\phi)}$ functions exhibiting a late time approach to a constant positive asymptote.

early times, the $c_{3,4}$ contributions grow relative to this and at some redshift they become comparable. This leads to cancellations in the denominator of $G_{\text{eff}}^{(\phi)}$ and account for the observed bump. The height and redshift of this feature depend on the values of c_{2-5} and also $\rho_\pi(z_i)$; for example for lower initial π density, the hierarchy between the c_{3-5} terms is less pronounced, leading to a feature at earlier times. Generically, then, the paths of gravity in Galileon gravity are quite intricate and rich.

VI. CONCLUSIONS

Galileon gravity is a well defined theory with a rich phenomenology. We explored the predictions for expansion history giving rise to current acceleration, despite the absence of any cosmological constant or potential, and found early time scaling and tracing solutions and late time de Sitter states. Equally important is the behavior of the inhomogeneities in the Galileon field, and we discussed requirements for a sound theory in terms of the no-ghost condition, Laplace stability, and positivity of energy density. By creating a large ensemble of expansion histories, we were able to use these theoretical conditions to efficiently rule out large regions of the multi dimensional parameter space. In this work we focussed solely on the dynamics of scalar perturbations; if we include tensor modes then we expect even more stringent bounds on the Galileon parameters (see for example [53]). Confronting the model with data to further constrain the parameters will be the subject of future work.

The modifications to gravity induced by the Galileon terms can be characterized by gravitational strengths (effective Newton's constants) entering the Poisson equations for the various combinations of metric potentials, $G_{\text{eff}}^{(\phi)}$, $G_{\text{eff}}^{(\psi)}$, and $G_{\text{eff}}^{(\phi+\psi)}$. We derived expressions for these and solutions in the early time and late time limits. Considering special cases and general scans over the parameter space, we presented the “paths of gravity”, the evolution of the effective gravitational strengths over cosmic history.

Galileon models do not have simple, monotonic evolution of gravity. One special feature discovered is that in the de Sitter limit the “matter gravity” $G_{\text{eff}}^{(\psi)}$ entering into growth of structure and the “light gravity” $G_{\text{eff}}^{(\phi+\psi)}$ entering into light deflection are identical (unlike in many other theories), although generally different from general relativity. Gravity can suddenly become stronger at some epoch in $1 \lesssim z \lesssim 1000$, and then be restored to general relativity; it is intriguing to consider the possible effects on structure formation such as enabling early formation of massive clusters. The observed phenomenology of $G_{\text{eff}}^{(\psi)}$ at intermediate redshifts will significantly increase the (linear) power of the matter perturbations at late times, and this will be used to severely constrain the uncoupled Galileon in an upcoming study.

Models which include a coupling between the Galileon field and the energy-momentum tensor, or equivalently the field and the Ricci scalar in the case of linear coupling or the field kinetic term and the Einstein tensor in the case of derivative coupling, are severely constrained by conditions for soundness of the theory. In particular, we find that

each coupling term must be subdominant to uncoupled Galileon terms in the early universe, and generally small. Deviations from the uncoupled case therefore only tend to show up at late times. For example, the linearly coupled case does not have a true de Sitter fixed point.

Galileon gravity offers a rich variety in both the expansion behavior (e.g. equations of state that can track the matter component and give rise to early dark energy) and gravity behavior (spikes of deviations from, and restoration to, general relativity). It provides a theoretically viable alternative to Einstein gravity, motivated by higher dimension geometric theories and protected by symmetries, without many of the naturalness issues of the cosmological constant or scalar field potentials. Galileon cosmology's implications for observations at high and low redshift make it an excellent theory to explore, and to test as an origin for cosmic acceleration and an extension to Einstein gravity.

Acknowledgments

This work has been supported by World Class University grant R32-2009-000-10130-0 through the National Research Foundation, Ministry of Education, Science and Technology of Korea, and in part by the Director, Office of Science, Office of High Energy Physics, of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231. We thank Claudia de Rham, Giulia Gubitosi, Andrew Tolley and Michael Kopp for helpful discussions.

Appendix A: Jordan Frame

In the main body of the paper we choose to work in a frame in which there is no direct coupling between the π field and standard matter, instead the coupling is exhibited directly to the metric quantities. To accomplish this we here consider the transform between the Einstein and Jordan frames in the weak field limit (see also [42]). We consider two couplings throughout this work, and we treat them separately here. Beginning with the linear coupling, we write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and the Lagrangian to quadratic order in $h_{\mu\nu}$ is given by

$$\mathcal{L} = -\frac{M_{\text{pl}}^2}{2} h^{\mu\nu} \delta G_{\mu\nu} + \frac{1}{2} \sum_{i=1}^5 c_i \mathcal{L}_i + \frac{1}{2} h^{\mu\nu} T_{\mu\nu} + \frac{c_0}{M_{\text{pl}}} \pi T, \quad (\text{A1})$$

where $\delta G_{\mu\nu}$ is the linearized Einstein tensor and $T = \eta^{\mu\nu} T_{\mu\nu}$, where $T_{\mu\nu}$ is the energy momentum tensor of matter. In this linearized limit we can perform the transformation $h_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} = h_{\mu\nu} + 2c_0\pi/M_{\text{pl}}\eta_{\mu\nu}$ to remove the c_0 coupling term. Under this transformation, neglecting any boundary terms, the linearized Einstein contribution in Eq. (A1), the first term on the right hand side, introduces three additional terms, two of which can be absorbed into the $c_{2,3}$ parameters. In terms of $\bar{h}_{\mu\nu}$ and π , the Lagrangian can be written as

$$\mathcal{L} = -\frac{M_{\text{pl}}^2}{2} \bar{h}^{\mu\nu} \delta \bar{G}_{\mu\nu} + 2c_0 M_{\text{pl}} \pi \eta^{\mu\nu} \delta \bar{G}_{\mu\nu} + \frac{1}{2} \sum_{i=1}^5 \bar{c}_i \mathcal{L}_i + \frac{1}{2} \bar{h}^{\mu\nu} T_{\mu\nu}, \quad (\text{A2})$$

where \bar{c}_i are the redefined constants, and $\delta \bar{G}_{\mu\nu}$ is the linearized Einstein tensor involving on $\bar{h}_{\mu\nu}$. Having constructed the linearized Jordan frame limit, we promote the action to its full, nonlinear form:

$$S = \int \sqrt{-g} d^4x \left[\left(1 - 2c_0 \frac{\pi}{M_{\text{pl}}} \right) \frac{M_{\text{pl}}^2 R}{2} - \frac{c_2}{2} (\partial\pi)^2 - \frac{c_3}{M^3} (\partial\pi)^2 \square\pi - \frac{c_4 \mathcal{L}_4}{2} - \frac{c_5 \mathcal{L}_5}{2} - \mathcal{L}_m \right]. \quad (\text{A3})$$

Similarly, we can repeat the above procedure with the derivative coupling $c_G T^{\mu\nu} \partial_\mu \pi \partial_\nu \pi$. Specifically, we write the linearized Lagrangian as

$$\mathcal{L} = -\frac{M_{\text{pl}}^2}{2} h^{\mu\nu} \delta G_{\mu\nu} + \frac{1}{2} \sum_{i=1}^5 c_i \mathcal{L}_i + \frac{1}{2} h^{\mu\nu} T_{\mu\nu} + \frac{c_G}{M_{\text{pl}} M^3} T_{\mu\nu} \partial^\mu \pi \partial^\nu \pi. \quad (\text{A4})$$

Performing the transformation $h_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} = h_{\mu\nu} + 2(c_G/M_{\text{pl}}M^3)\partial_\mu\pi\partial_\nu\pi$, we remove the explicit coupling to matter at the expense of introducing additional mixing terms between $\bar{h}_{\mu\nu}$ and π . The transform introduces a term that can be absorbed into c_4 , and a new term of the form $\partial^\mu\pi\partial^\nu\pi\delta G_{\mu\nu}$. We find

$$\mathcal{L} = -\frac{M_{\text{pl}}^2}{2} \bar{h}^{\mu\nu} \delta \bar{G}_{\mu\nu} + 2c_G \frac{M_{\text{pl}}}{M^3} \partial^\mu \pi \partial^\nu \pi \delta \bar{G}_{\mu\nu} + \frac{1}{2} \sum_{i=1}^5 \bar{c}_i \mathcal{L}_i + \frac{1}{2} \bar{h}^{\mu\nu} T_{\mu\nu}. \quad (\text{A5})$$

Promoting to a full, nonlinear action, we arrive at

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{pl}}^2 R}{2} - \frac{c_2}{2} (\partial\pi)^2 - \frac{c_3}{M^3} (\partial\pi)^2 \square\pi - \frac{c_4 \mathcal{L}_4}{2} - \frac{c_5 \mathcal{L}_5}{2} - \frac{M_{\text{pl}}}{M^3} c_{\text{G}} G^{\mu\nu} \partial_\mu \pi \partial_\nu \pi - \mathcal{L}_{\text{m}} \right]. \quad (\text{A6})$$

In regions of high curvature, the equivalence between the Jordan and Einstein frames considered here will no longer be valid, however we work with the Jordan frame quantities throughout.

Appendix B: Field Equations

To obtain the Einstein and scalar field equations we vary the action of Eq. (6) with respect to the metric $g_{\mu\nu}$ and Galileon field π , obtaining

$$\begin{aligned} & \frac{M^3 c_2}{2} \square\pi + c_3 (\square\pi)^2 - c_3 (\nabla_\mu \nabla_\nu \pi)^2 - c_3 R^{\mu\nu} \nabla_\mu \pi \nabla_\nu \pi + M_{\text{pl}} c_{\text{G}} G^{\mu\nu} \nabla_\mu \nabla_\nu \pi \\ &= \frac{M_{\text{pl}} M^3 c_0}{2} R + \frac{c_4}{4M^3} \left[-4(\square\pi)^3 - 8\nabla_\alpha \nabla_\nu \pi \nabla^\nu \nabla^\lambda \pi \nabla_\lambda \nabla^\alpha \pi + 12(\square\pi) \nabla_\alpha \nabla_\beta \pi \nabla^\alpha \nabla^\beta \pi + 2\square\pi \nabla_\alpha \pi \nabla^\alpha \pi R \right. \\ & \quad + 4\nabla_\alpha \pi \nabla_\beta \pi \nabla^\alpha \nabla^\beta \pi R + 8(\square\pi) R^{\mu\nu} \nabla_\mu \pi \nabla_\nu \pi - 4\nabla_\alpha \pi \nabla^\alpha \pi \nabla^\lambda \nabla^\sigma \pi R_{\lambda\sigma} - 16\nabla_\alpha \pi \nabla^\alpha \nabla^\sigma \pi R_{\sigma\rho} \nabla^\rho \pi \\ & \quad \left. - 8\nabla_\alpha \pi \nabla_\beta \pi \nabla_\rho \nabla_\sigma \pi R^{\alpha\rho\beta\sigma} \right] \\ & \left(1 - 2c_0 \frac{\pi}{M_{\text{pl}}} \right) M_{\text{pl}}^2 G_{\mu\nu} = T_{\mu\nu} - 2M_{\text{pl}} c_0 (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) \pi + T_{\mu\nu}^{(c_2)} + T_{\mu\nu}^{(c_3)} + T_{\mu\nu}^{(c_4)} + T_{\mu\nu}^{(c_5)} + T_{\mu\nu}^{(c_{\text{G}})} \end{aligned}$$

where the c_{2-5} contributions to the Galileon energy momentum tensor, $T_{\mu\nu}^{(c_{2-5})}$, were first calculated in [2]. We repeat them here for completeness, and add the c_{G} contribution.

$$T_{\mu\nu}^{(c_2)} = c_2 \left[\nabla_\mu \pi \nabla_\nu \pi - \frac{1}{2} g_{\mu\nu} (\partial \pi)^2 \right] \quad (B1)$$

$$T_{\mu\nu}^{(c_3)} = \frac{c_3}{M^3} [2 \nabla_\mu \pi \nabla_\nu \pi \square \pi + g_{\mu\nu} \nabla_\alpha \pi \nabla^\alpha (\partial \pi)^2 - 2 \nabla_{(\mu} \pi \nabla_{\nu)} (\partial \pi)^2] \quad (B2)$$

$$\begin{aligned} T_{\mu\nu}^{(c_4)} = & -\frac{c_4}{M^6} [4 \square \pi \nabla_\alpha \pi \nabla_\mu \pi \nabla^\alpha \nabla_\nu \pi + 4 \square \pi \nabla_\alpha \pi \nabla_\nu \pi \nabla^\alpha \nabla_\mu \pi - 2 \square \pi \square \pi \nabla_\mu \pi \nabla_\nu \pi + 2 \square \pi \nabla_\alpha \pi \nabla^\alpha \pi \nabla_\mu \nabla_\nu \pi \\ & + 4 \nabla^\alpha \pi \nabla^\lambda \pi \nabla_\alpha \nabla_\lambda \pi \nabla_\mu \nabla_\nu \pi - 4 \nabla^\alpha \pi \nabla^\lambda \pi \nabla_\alpha \nabla_\mu \pi \nabla_\lambda \nabla_\nu \pi + 2 \nabla_\mu \pi \nabla_\nu \pi \nabla^\alpha \nabla^\beta \pi \nabla_\alpha \nabla_\beta \pi \\ & - 2 \nabla_\alpha \pi \nabla^\alpha \pi \nabla_\mu \nabla_\beta \pi \nabla_\nu \nabla^\beta \pi - 4 \nabla_\alpha \pi \nabla_\nu \pi \nabla^\alpha \nabla_\sigma \pi \nabla^\sigma \nabla_\mu \pi - 4 \nabla_\alpha \pi \nabla_\mu \pi \nabla^\alpha \nabla_\sigma \pi \nabla^\sigma \nabla_\nu \pi - g_{\mu\nu} (\square \pi)^2 \nabla_\alpha \pi \nabla^\alpha \pi \\ & - 4 g_{\mu\nu} \square \pi \nabla_\alpha \pi \nabla_\lambda \pi \nabla^\alpha \nabla^\lambda \pi + 4 g_{\mu\nu} \nabla_\alpha \pi \nabla_\omega \pi \nabla^\alpha \nabla_\lambda \pi \nabla^\lambda \nabla^\omega \pi + g_{\mu\nu} \nabla_\alpha \pi \nabla^\alpha \pi \nabla_\lambda \nabla_\sigma \pi \nabla^\lambda \nabla^\sigma \pi \\ & + \nabla_\mu \pi \nabla_\nu \pi \nabla_\alpha \pi \nabla^\alpha \pi R - \frac{g_{\mu\nu}}{4} \nabla_\alpha \pi \nabla^\alpha \pi \nabla_\lambda \pi \nabla^\lambda \pi R - 2 \nabla_\alpha \pi \nabla^\alpha \pi R_{\lambda\mu} \nabla_\nu \pi \nabla^\lambda \pi - 2 \nabla_\alpha \pi \nabla^\alpha \pi R_{\lambda\nu} \nabla_\mu \pi \nabla^\lambda \pi \\ & + \frac{1}{2} R_{\mu\nu} \nabla_\alpha \pi \nabla^\alpha \pi \nabla_\lambda \pi \nabla^\lambda \pi + 2 g_{\mu\nu} R_{\sigma\omega} \nabla^\sigma \pi \nabla^\omega \pi \nabla_\alpha \pi \nabla^\alpha \pi - 2 \nabla_\alpha \pi \nabla^\alpha \pi g_{\mu\zeta} R^\zeta_{\sigma\nu\omega} \nabla^\sigma \pi \nabla^\omega \pi] \end{aligned} \quad (B3)$$

$$\begin{aligned} T_{\mu\nu}^{(c_5)} = & \frac{c_5}{M^9} \left[(\square \pi)^3 \nabla_\mu \pi \nabla_\nu \pi + (\square \pi)^3 \nabla_\alpha \pi \nabla^\alpha \pi g_{\mu\nu} - 3 (\square \pi)^2 \nabla_\alpha \pi \nabla^\alpha \pi \nabla_\mu \nabla_\nu \pi \right. \\ & - 3 (\square \pi)^2 \nabla_\rho \pi (\nabla^\rho \nabla_\mu \pi \nabla_\nu \pi + \nabla^\rho \nabla_\nu \pi \nabla_\mu \pi) + 3 (\square \pi)^2 \nabla_\alpha \pi \nabla_\beta \pi \nabla^\alpha \nabla^\beta \pi g_{\mu\nu} \\ & + 6 (\square \pi) \nabla_\alpha \pi \nabla^\alpha \pi \nabla_\mu \nabla_\sigma \pi \nabla^\sigma \nabla_\nu \pi - 6 (\square \pi) \nabla^\alpha \pi \nabla^\beta \pi \nabla_\alpha \nabla_\beta \pi \nabla_\mu \nabla_\nu \pi - 3 (\square \pi) \nabla^\alpha \nabla^\beta \pi \nabla_\alpha \nabla_\beta \pi \nabla_\mu \nabla_\nu \pi \\ & + 6 (\square \pi) \nabla^\alpha \pi \nabla^\beta \pi \nabla_\alpha \nabla_\beta \nabla_\mu \pi \nabla_\nu \pi + 6 (\square \pi) \nabla^\alpha \pi \nabla_\alpha \nabla_\beta \pi (\nabla^\beta \nabla_\mu \pi \nabla_\nu \pi + \nabla^\beta \nabla_\nu \pi \nabla_\mu \pi) \\ & - 3 (\square \pi) \nabla_\alpha \pi \nabla^\alpha \pi \nabla_\lambda \nabla_\sigma \pi \nabla^\lambda \nabla^\sigma \pi g_{\mu\nu} - 6 (\square \pi) \nabla^\alpha \pi \nabla^\beta \pi \nabla_\alpha \nabla_\sigma \nabla_\beta \nabla^\sigma \pi g_{\mu\nu} - \frac{3}{2} (\square \pi) \nabla_\alpha \pi \nabla^\alpha \pi \nabla_\mu \pi \nabla_\nu \pi R \\ & + 3 (\square \pi) \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\beta \pi (R_{\beta\mu} \nabla_\nu \pi + R_{\beta\nu} \nabla_\mu \pi) - 3 (\square \pi) \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\lambda \pi \nabla^\beta \pi R_{\lambda\beta} g_{\mu\nu} \\ & + 3 (\square \pi) \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\lambda \pi \nabla^\sigma \pi R_{\mu\sigma\nu\lambda} + 3 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\lambda \nabla^\sigma \pi \nabla_\lambda \nabla_\sigma \pi \nabla_\mu \nabla_\nu \pi - 6 \nabla_\alpha \pi \nabla^\alpha \pi \nabla_\mu \nabla^\sigma \pi \nabla_\sigma \nabla_\lambda \pi \nabla^\lambda \nabla_\nu \pi \\ & + 6 \nabla^\alpha \pi \nabla^\beta \pi \nabla_\alpha \nabla_\beta \pi \nabla^\lambda \nabla_\mu \pi \nabla_\lambda \nabla_\nu \pi + 6 \nabla^\alpha \pi \nabla^\beta \pi \nabla_\alpha \nabla_\sigma \pi \nabla^\sigma \nabla_\beta \nabla_\mu \nabla_\nu \pi + 2 \nabla_\rho \nabla^\sigma \pi \nabla_\sigma \nabla^\lambda \pi \nabla_\lambda \nabla^\rho \pi \nabla_\mu \pi \nabla_\nu \pi \\ & + 3 \nabla^\alpha \nabla^\beta \pi \nabla_\alpha \nabla_\beta \pi \nabla_\lambda \pi (\nabla^\lambda \nabla_\mu \pi \nabla_\nu \pi + \nabla^\lambda \nabla_\nu \pi \nabla_\mu \pi) - 6 \nabla_\rho \pi \nabla^\rho \nabla^\sigma \pi \nabla_\sigma \nabla_\lambda \pi (\nabla^\lambda \nabla_\mu \pi \nabla_\nu \pi + \nabla^\lambda \nabla_\nu \pi \nabla_\mu \pi) \\ & - 6 \nabla^\rho \pi \nabla_\rho \nabla_\lambda \pi \nabla_\sigma (\nabla^\lambda \nabla_\mu \pi \nabla^\sigma \nabla_\nu \pi + \nabla^\lambda \nabla_\nu \pi \nabla^\sigma \nabla_\mu \pi) + 2 \nabla_\alpha \pi \nabla^\alpha \pi \nabla_\sigma \nabla^\lambda \pi \nabla_\lambda \nabla^\kappa \pi \nabla_\kappa \nabla^\sigma \pi g_{\mu\nu} \\ & - 3 \nabla^\alpha \pi \nabla^\beta \pi \nabla_\alpha \nabla_\beta \pi \nabla^\lambda \nabla^\sigma \pi \nabla_\lambda \nabla_\sigma \pi g_{\mu\nu} + 6 \nabla^\rho \pi \nabla^\kappa \pi \nabla_\rho \nabla_\sigma \pi \nabla^\sigma \nabla_\lambda \nabla^\lambda \nabla_\kappa \pi g_{\mu\nu} \\ & + \frac{3}{2} \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\sigma \pi (\nabla_\sigma \nabla_\mu \pi \nabla_\nu \pi + \nabla_\sigma \nabla_\nu \pi \nabla_\mu \pi) R - \frac{3}{2} \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\lambda \pi \nabla^\sigma \pi \nabla_\lambda \nabla_\sigma \pi R g_{\mu\nu} \\ & + 3 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\lambda \pi \nabla^\sigma \pi \nabla_\lambda \nabla_\sigma \pi R_{\mu\nu} + 3 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\lambda \pi \nabla^\sigma \pi R_{\lambda\sigma} \nabla_\mu \nabla_\nu \pi + 3 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\lambda \pi \nabla^\sigma \pi R_{\lambda\sigma} \nabla_\mu \pi \nabla_\nu \pi \\ & - 3 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\lambda \pi \nabla_\lambda \nabla^\sigma \pi (R_{\sigma\mu} \nabla_\nu \pi + R_{\sigma\nu} \nabla_\mu \pi) - 3 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\lambda \pi \nabla^\sigma \pi (R_{\lambda\mu} \nabla_\nu \nabla_\sigma \pi + R_{\lambda\nu} \nabla_\mu \nabla_\sigma \pi) \\ & - 3 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\sigma \pi R_{\sigma\lambda} (\nabla^\lambda \nabla_\mu \pi \nabla_\nu \pi + \nabla^\lambda \nabla_\nu \pi \nabla_\mu \pi) + 6 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\sigma \pi \nabla_\sigma \nabla^\lambda \pi R_{\lambda\kappa} \nabla^\kappa \pi g_{\mu\nu} \\ & - 3 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\sigma \pi \nabla_\lambda \nabla^\lambda \pi (R_{\mu\lambda\sigma\kappa} \nabla_\nu \pi + R_{\nu\lambda\sigma\kappa} \nabla_\mu \pi) + 3 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\sigma \pi \nabla^\lambda \pi \nabla^\lambda \pi (R_{\mu\sigma\lambda\kappa} \nabla^\kappa \nabla_\nu \pi + R_{\nu\sigma\lambda\kappa} \nabla^\kappa \nabla_\mu \pi) \\ & \left. - 3 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\sigma \pi \nabla_\sigma \nabla^\lambda \pi \nabla^\kappa \pi (R_{\mu\lambda\nu\kappa} + R_{\nu\lambda\mu\kappa}) + 3 \nabla_\alpha \pi \nabla^\alpha \pi \nabla^\sigma \pi \nabla^\lambda \pi \nabla^\kappa \nabla^\tau \pi R_{\sigma\kappa\lambda\tau} g_{\mu\nu} \right] \quad (B4)$$

$$\begin{aligned} T_{\mu\nu}^{(c_G)} = & \frac{M_{\text{Pl}}}{M^3} c_G [g_{\mu\nu} \square \pi \square \pi - 2 \square \pi \nabla_\mu \nabla_\nu \pi + 2 \nabla_\mu \nabla_\lambda \pi \nabla_\nu \nabla^\lambda \pi - g_{\mu\nu} \nabla_\lambda \nabla_\sigma \pi \nabla^\lambda \nabla^\sigma \pi] \\ & - \frac{M_{\text{Pl}}}{M^3} c_G \left[R_{\mu\nu} \nabla_\alpha \pi \nabla^\alpha \pi + R \nabla_\mu \pi \nabla_\nu \pi - \frac{1}{2} g_{\mu\nu} R \nabla_\alpha \pi \nabla^\alpha \pi \right] \\ & + 2 \frac{M_{\text{Pl}}}{M^3} c_G [R_{\lambda\nu} \nabla^\lambda \pi \nabla_\mu \pi + R_{\lambda\mu} \nabla^\lambda \pi \nabla_\nu \pi - g_{\mu\nu} R_{\rho\lambda} \nabla^\rho \pi \nabla^\lambda \pi + R^\sigma_{\mu\beta\nu} \nabla^\beta \pi \nabla_\sigma \pi] . \end{aligned} \quad (B5)$$

Appendix C: Perturbed Field Equations

To obtain the perturbed field equations in the subhorizon limit, taking the Newtonian gauge and defining $\delta y = \delta \pi / M_{\text{Pl}}$, we linearize the perturbed $(i, j \neq i)$ and $(0, 0)$ Einstein, π dynamical, and fluid conservation equations in δy ,

the metric potentials ϕ and ψ , and the density perturbation δ_m . We obtain the following equations

$$\begin{aligned} -[1 - 2c_0y] (\bar{\nabla}^2\psi - \bar{\nabla}^2\phi) &= -2c_0\bar{\nabla}^2\delta y \\ &- c_4 \left[6\bar{H}^3x^2 \left(\bar{H}'x + \bar{H}x' + \frac{1}{3}\bar{H}x \right) \bar{\nabla}^2\delta y + \frac{\bar{H}^4x^4}{2} \bar{\nabla}^2\phi + \frac{3}{2}\bar{H}^4x^4\bar{\nabla}^2\psi \right] \\ &+ c_G [2(\bar{H}\bar{H}'x + \bar{H}^2x' + \bar{H}^2x)\bar{\nabla}^2\delta y + \bar{H}^2x^2\bar{\nabla}^2\phi + \bar{H}^2x^2\bar{\nabla}^2\psi] \\ &+ c_5 [3\bar{H}^6x^5\bar{\nabla}^2\psi - 3\bar{H}^5x^4(\bar{H}x' + \bar{H}'x)\bar{\nabla}^2\phi + \bar{H}^5x^3(12\bar{H}x' + 15\bar{H}'x + 3\bar{H}x)\bar{\nabla}^2\delta y] \end{aligned} \quad (C1)$$

$$\begin{aligned} \frac{2}{a^2} [1 - 2c_0y] \bar{\nabla}^2\phi &= \frac{\rho_m}{H_0^2 M_{\text{pl}}^2} \delta_m + \left(2c_3\bar{H}^2\frac{x^2}{a^2} - \frac{2c_0}{a^2} \right) \bar{\nabla}^2\delta y - c_4 \left(\frac{12\bar{H}^4x^3}{a^2} \bar{\nabla}^2\delta y - \frac{3\bar{H}^4x^4}{a^2} \bar{\nabla}^2\phi \right) \\ &+ c_G \left[4\bar{H}^2\frac{x}{a^2} \bar{\nabla}^2\delta y - 2\bar{H}^2\frac{x^2}{a^2} \bar{\nabla}^2\phi \right] + c_5 \left[\frac{15\bar{H}^6x^4}{a^2} \bar{\nabla}^2\delta y - \frac{6\bar{H}^6x^5}{a^2} \bar{\nabla}^2\phi \right] \end{aligned} \quad (C2)$$

$$\begin{aligned} -2 \left[c_3\bar{H}\bar{H}'x + c_3\bar{H}^2x' + 2c_3\bar{H}^2x - \frac{c_2}{4} \right] \bar{\nabla}^2\delta y - c_3\bar{H}^2x^2\bar{\nabla}^2\psi &= -c_0 [\bar{\nabla}^2\psi - 2\bar{\nabla}^2\phi] + c_4 [-6\bar{H}^4x^3\bar{\nabla}^2\psi \\ &+ (6\bar{H}^3x^2(\bar{H}x' + x\bar{H}') + 2\bar{H}^4x^3)\bar{\nabla}^2\phi - (12\bar{H}^3x(\bar{H}x' + x\bar{H}') + 13\bar{H}^4x^2 + 6\bar{H}^3x^2\bar{H}')\bar{\nabla}^2\delta y] \\ &+ c_G (2\bar{H}\bar{H}' + 3\bar{H}^2)\bar{\nabla}^2\delta y + 2c_G\bar{H}^2x\bar{\nabla}^2\psi - 2c_G(\bar{H}(\bar{H}x' + \bar{H}'x) + \bar{H}^2x)\bar{\nabla}^2\phi \\ &+ c_5 \left[(18\bar{H}^6x^2x' + 30\bar{H}^5x^3\bar{H}' + 12\bar{H}^6x^3)\bar{\nabla}^2\delta y + \frac{15}{2}\bar{H}^6x^4\bar{\nabla}^2\psi - (3\bar{H}^6x^4 + 15\bar{H}^5x^4\bar{H}' + 12\bar{H}^6x^3x')\bar{\nabla}^2\phi \right] \end{aligned} \quad (C3)$$

$$\bar{H}^2\delta_m'' + \bar{H}\bar{H}'\delta_m' + 2\bar{H}^2\delta_m' = \frac{1}{a^2}\bar{\nabla}^2\psi \quad (C4)$$

where $\bar{\nabla} = \nabla/H_0$, ρ_m is the matter density, and $\delta_m = \delta\rho_m/\rho_m$. We have assumed that the quasistatic approximation holds at all times: $\delta\ddot{y}, \ddot{\phi}, \ddot{\psi} \ll \bar{\nabla}^2\delta y, \bar{\nabla}^2\phi, \bar{\nabla}^2\psi$. Note that in Eq. (C1) the “slip” between ϕ and ψ is not sourced by c_2 or c_3 . The above equations are in agreement with those derived in [30] in the uncoupled case $c_0 = c_G = 0$.

Equations (C1) and (C3) can be used to eliminate $\bar{\nabla}^2\delta y$ and $\bar{\nabla}^2\psi$ from the Poisson equation (C2). Hence we can describe the effect of the Galileon on subhorizon density perturbations through two functions of the background quantities \bar{H}, x, y . We define these functions $G_{\text{eff}}^{(\phi)}$ and $G_{\text{eff}}^{(\psi)}$ in Eqs. (24) and (25), and $G_{\text{eff}}^{(\phi+\psi)} = [G_{\text{eff}}^{(\phi)} + G_{\text{eff}}^{(\psi)}]/2$.

Appendix D: No-Ghost Condition in Detail

For a given coupling, the action (6) describes a model with five free parameters: the magnitude of the coupling strength and the kinetic terms (c_2, c_3, c_4, c_5). However, one must be careful to restrict our analysis to the parameter space in which the model is theoretically viable. Of particular importance is the absence of propagating ghost degrees of freedom, that is degrees of freedom whose kinetic contribution to the Hamiltonian is negative, making it unbounded from below.

In [9] a “no-ghost” condition was derived for models of the form

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2}F(\pi)R + \frac{1}{2}f_2(\pi, X) + \zeta(\pi)(\partial\pi)^2\Box\pi \right], \quad (D1)$$

where $X = -g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi/2$, and $F(\pi)$, $f_2(\pi, X)$ and $\zeta(\pi)$ are arbitrary functions. The no-ghost condition corresponds to [9]

$$(24\zeta H\dot{\pi} - 8\zeta_{,\pi}\dot{\pi}^2 + f_{2,X} + f_{2,XX}\dot{\pi}^2)F\dot{\pi}^2 + 3(\dot{F} - 2\zeta\dot{\pi}^3)^2 > 0 \quad (D2)$$

The action (D1) is equivalent to ours with $c_4 = c_5 = c_0 = c_G = 0$; in this case we can use the above condition as a constraint on our parameter space. When we introduce the couplings $c_{0,G}$ and more general kinetic terms $c_{4,5}$ the analysis must be redone as follows.

To construct the no-ghost condition, we utilise four equations: the π equation of motion and the $(0,0)$, trace, and $(i, j \neq i)$ Einstein equations. To deduce whether any propagating degrees of freedom exhibit ghost behaviour, it suffices to look for the sign of the coefficient of the $\ddot{\phi}$ and $\delta\ddot{y}$ terms. Our approach is to note that the $(i, j \neq i)$ equation gives an algebraic relation between δy , ψ and ϕ , and the $(0,0)$ Einstein equation contains only $\nabla^2\psi$, $\nabla^2\phi$

and $\nabla^2\delta y$ terms (and first order time derivatives, which are not important in what follows). Therefore combining these equations yields a relationship between $\nabla^2\phi$ and $\nabla^2\delta y$. The π equation of motion and trace Einstein equation contain the terms $\delta\ddot{y}$, $\ddot{\phi}$, $\nabla^2\psi$, $\nabla^2\phi$ and $\nabla^2\delta y$, and therefore by eliminating $\nabla^2\psi$ and using our relationship between $\nabla^2\delta y$ and $\nabla^2\phi$ we can construct equations of the form

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{B}\nabla^2\mathbf{x} = \mathbf{C} . \quad (\text{D3})$$

Here \mathbf{A} , \mathbf{B} are diagonal 2×2 matrices that are functions of background quantities such as $\dot{\pi}$, H etc. and \mathbf{x} is the two dimensional vector containing δy and ϕ . The matrix \mathbf{C} contains lower order time and spatial derivatives of \mathbf{x} and is unimportant in calculating the no-ghost condition. For the simple case of a minimally coupled, canonical scalar field containing the c_2 term only, the no-ghost condition corresponds to $c_2 > 0$, and therefore we simply need to ensure that \mathbf{A} preserves the same sign conventions.

We begin with a simple case taking $c_{2,3}$ only. The $(i, j \neq i)$ equation gives $\psi = \phi$, and the $(0, 0)$ component of the Einstein equations yields the relationship

$$\nabla^2\phi = \frac{c_3}{M_{\text{pl}}^2 M^3} (\dot{\pi})^2 \nabla^2\delta y + \dots \quad (\text{D4})$$

which will not be necessary in what follows. The \dots denote lower derivative contributions to the equations that will not be relevant to the stability condition derived here.

The trace and π equations are

$$(6\ddot{\phi} - 2\nabla^2\phi) = \frac{c_3}{M_{\text{pl}} M^3} (6\dot{\pi}^2 \delta\ddot{y} - 2\dot{\pi}^2 \nabla^2\delta y) + \dots \quad (\text{D5})$$

$$\frac{M^3 c_2}{2} (-\delta\ddot{y} + \nabla^2\delta y) + 6c_3 H \dot{\pi} \delta\ddot{y} - 2c_3 (\dot{\pi} + 2H\dot{\pi}) \nabla^2\delta y - c_3 \dot{\pi}^2 (3\ddot{\phi} - \nabla^2\phi) + \dots = 0 \quad (\text{D6})$$

Eliminating ϕ , we find that the coefficient of the $\delta\ddot{y}$ term is given by

$$-\frac{M^3 c_2}{2} + 6c_3 H \dot{\pi} - 3c_3^2 \frac{\dot{\pi}^4}{M_{\text{pl}}^2 M^3} < 0 \quad (\text{D7})$$

which must be less than zero to ensure that perturbations of the π field have the correct sign. This condition is in agreement with Eq. (D2).

We now expand our approach to include the $c_{4,5,G}$ terms in the action. Performing exactly the same steps as above, we find the following inequality for the no-ghost condition

$$-\frac{c_2}{2} + 6c_3 \bar{H}^2 x + 3c_G \bar{H}^2 - 27c_4 \bar{H}^4 x^2 + 30c_5 \bar{H}^6 x^3 + 2 \frac{(3c_3 \bar{H}^2 x^2 + 6c_G \bar{H}^2 x - 18c_4 \bar{H}^4 x^3 + \frac{45}{2} c_5 \bar{H}^6 x^4 - 3c_0)^2}{-6(1 - 2c_0 y) - 6c_G \bar{H}^2 x^2 + 9c_4 \bar{H}^4 x^4 - 18c_5 \bar{H}^6 x^5} < 0 \quad (\text{D8})$$

This must be satisfied at all times during the cosmological evolution to ensure that at the level of linear perturbations around the cosmological background $\delta\pi$ has a kinetic term that contributes positively to the Hamiltonian. Note that the no-ghost condition (D8) can be written in terms of the κ_i functions as

$$\kappa_2 + \frac{3}{2} \frac{\kappa_5^2}{\kappa_4} < 0 . \quad (\text{D9})$$

Returning to our solutions for the linear coupling case, Eqs. (45)-(47) valid during matter domination, we can show that for parameters that give a positive energy density $\rho_\pi > 0$, the condition (D8) is not satisfied. To see this, we note that at early times the no-ghost condition is well approximated by the first term κ_2 ,

$$-\frac{c_2}{2} + 6c_3 \bar{H}^2 x + 3c_G \bar{H}^2 - 27c_4 \bar{H}^4 x^2 + 30c_5 \bar{H}^6 x^3 < 0 . \quad (\text{D10})$$

Consider the various dominating pairs, starting with c_5, c_0 . To ensure that $\rho_\pi > 0$, we are forced to take $c_0 > 0$ as A_0 must be real. This in turn forces $c_5 > 0$, which violates Eq. (D10). We find the same dilemma for the c_4 solution and so on until we reach c_2 . Note that we have not shown that a ghost is generically present for the linearly coupled model whenever $\rho_\pi > 0$, but rather just for the particular solutions found in IV B. In fact during radiation domination the no-ghost condition is generically satisfied, as in the uncoupled case.

The no-ghost condition derived here is applicable to linear perturbations on an FRW background. A more complete analysis should be undertaken in regimes where the non-linear nature of the derivative self couplings becomes significant.

Appendix E: Laplace Instability

In the previous section we considered the sign of the $\delta\ddot{y}$ term in the linearized perturbation equations, and how the no-ghost condition can be used to place constraints on the Galileon parameters. In this section we consider the existence of a second instability that must also be avoided – the Laplace instability [29] (see also [57] for a very complete discussion of instabilities). This is a condition on the sound speed of the π field perturbation, obtained by linearizing the Einstein and π equations and eliminating the metric potentials to obtain a wave-like equation for $\delta\pi$. Specifically, we combine the $(0,0)$, trace, and $(i,j \neq i)$ Einstein equations with the perturbed π equation of motion, keeping all terms containing second order time and spatial derivatives of the fields. We find the following expression for $\delta\pi$:

$$\left(\kappa_2 + \frac{3}{2} \frac{\kappa_5^2}{\kappa_4}\right) \delta\ddot{\pi} + \left(\frac{2\kappa_3\kappa_5^2 + 2\kappa_4^2\kappa_6 - 4\kappa_1\kappa_4\kappa_5}{2\kappa_4^2}\right) \nabla^2 \delta\pi = \dots \quad (\text{E1})$$

Again, \dots represents terms unimportant to the stability argument. The negative definiteness of the $\delta\ddot{\pi}$ coefficient is the no-ghost condition, and the negative definiteness of the ratio of the $\nabla^2 \delta\pi$ and $\delta\ddot{\pi}$ coefficients is the Laplacian stability condition. This ratio is the negative of the sound speed squared,

$$c_s^2 \equiv \frac{4\kappa_1\kappa_4\kappa_5 - 2\kappa_3\kappa_5^2 - 2\kappa_4^2\kappa_6}{\kappa_4(2\kappa_4\kappa_2 + 3\kappa_5^2)}. \quad (\text{E2})$$

The Laplace stability condition is $c_s^2 \geq 0$. If this condition is violated, imaginary frequency behaviour would follow, leading to exponential growth of the π field perturbation. (It is conceivable that such behaviour is benign if the timescale associated with the growth is suitably large, however we opt for a conservative approach and enforce $c_s^2 \geq 0$ at all times during the cosmological evolution.)

At early times when $\Omega_\pi \ll 1$, the sound speed simplifies to $c_s^2 = -\kappa_6/\kappa_2$. At late times, for the uncoupled case we can derive a relatively straightforward asymptotic de Sitter form for the sound speed, corresponding to

$$\frac{(14 + 3C - 4D)(416 + 108C + 9C^2 - 32D - 16D^2)}{54(8 + 3C - 4D)(20 + 3C - 4D)} \geq 0 \quad (\text{E3})$$

where C and D are defined in Eqs. (62), (63).

In the main body of the text, we impose the positivity of ρ_π and the no-ghost and Laplace conditions for all $z < 500$. There is an additional condition that one might impose, that the $\delta\pi$ field must avoid “superluminal” behaviour $c_s^2 > 1$. However, if such a condition were violated it is not necessarily the case that the model is ruled out; see for example [58] for a detailed discussion of superluminal propagation in a different class of scalar fields. In addition, a more complete analysis should take into account the full perturbation equations when constructing the scalar field dispersion relation, including effective mass terms of the form $\bar{H}^2 \delta y$. For these reasons we note the potential existence of this constraint but do not impose it when scanning over the parameter space for viable models in Sec. V D.

[1] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D **79** (2009) 064036 [arXiv:0811.2197 [hep-th]].
[2] C. Deffayet, G. Esposito-Farese and A. Vikman, Phys. Rev. D **79** (2009) 084003 [arXiv:0901.1314 [hep-th]].
[3] C. Deffayet, S. Deser and G. Esposito-Farese, Phys. Rev. D **80** (2009) 064015 [arXiv:0906.1967 [gr-qc]].
[4] C. Deffayet, S. Deser and G. Esposito-Farese, Phys. Rev. D **82** (2010) 061501 [arXiv:1007.5278 [gr-qc]].
[5] A. I. Vainshtein, Phys. Lett. B **39** (1972) 393.
[6] C. Deffayet, G. R. Dvali, G. Gabadadze and A. I. Vainshtein, Phys. Rev. D **65** (2002) 044026 [hep-th/0106001].
[7] A. De Felice, R. Kase and S. Tsujikawa, arXiv:1111.5090 [gr-qc].
[8] C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saffin, arXiv:1106.2000 [hep-th].
[9] A. De Felice and S. Tsujikawa, JCAP **1007** (2010) 024 [arXiv:1005.0868 [astro-ph.CO]].
[10] A. De Felice, S. Mukohiyama and S. Tsujikawa, Phys. Rev. D **82** (2010) 023524 [arXiv:1006.0281 [astro-ph.CO]].
[11] D. F. Mota, M. Sandstad and T. Zlosnik, JHEP **1012** (2010) 051 [arXiv:1009.6151 [astro-ph.CO]].
[12] C. Deffayet, O. Pujolas, I. Sawicki and A. Vikman, JCAP **1010** (2010) 026 [arXiv:1008.0048 [hep-th]].
[13] T. Qiu, J. Evslin, Y. F. Cai, M. Li and X. Zhang, JCAP **1110** (2011) 036 [arXiv:1108.0593 [hep-th]].
[14] D. A. Easson, I. Sawicki and A. Vikman, JCAP **1111** (2011) 021 [arXiv:1109.1047 [hep-th]].
[15] O. Pujolas, I. Sawicki and A. Vikman, JHEP **1111** (2011) 156 [arXiv:1103.5360 [hep-th]].
[16] A. De Felice, T. Kobayashi and S. Tsujikawa, Phys. Lett. B **706** (2011) 123 [arXiv:1108.4242 [gr-qc]].
[17] K. Hinterbichler, M. Trodden and D. Wesley, Phys. Rev. D **82** (2010) 124018 [arXiv:1008.1305 [hep-th]].

- [18] M. Andrews, K. Hinterbichler, J. Khoury and M. Trodden, Phys. Rev. D **83** (2011) 044042 [arXiv:1008.4128 [hep-th]].
- [19] G. L. Goon, K. Hinterbichler and M. Trodden, Phys. Rev. D **83** (2011) 085015 [arXiv:1008.4580 [hep-th]].
- [20] A. Padilla, P. M. Saffin and S. -Y. Zhou, JHEP **1012** (2010) 031 [arXiv:1007.5424 [hep-th]].
- [21] A. Padilla, P. M. Saffin and S. -Y. Zhou, JHEP **1101** (2011) 099 [arXiv:1008.3312 [hep-th]].
- [22] A. Padilla, P. M. Saffin and S. -Y. Zhou, Phys. Rev. D **83** (2011) 045009 [arXiv:1008.0745 [hep-th]].
- [23] C. de Rham and A. J. Tolley, JCAP **1005** (2010) 015 [arXiv:1003.5917 [hep-th]].
- [24] C. de Rham and G. Gabadadze, Phys. Rev. D **82** (2010) 044020 [arXiv:1007.0443 [hep-th]].
- [25] C. Burrage, C. de Rham and L. Heisenberg, JCAP **1105** (2011) 025 [arXiv:1104.0155 [hep-th]].
- [26] F. P. Silva and K. Koyama, Phys. Rev. D **80** (2009) 121301 [arXiv:0909.4538 [astro-ph.CO]].
- [27] S. Mizuno and K. Koyama, Phys. Rev. D **82** (2010) 103518 [arXiv:1009.0677 [hep-th]].
- [28] T. Kobayashi, Phys. Rev. D **81** (2010) 103533 [arXiv:1003.3281 [astro-ph.CO]].
- [29] A. De Felice and S. Tsujikawa, arXiv:1008.4236 [hep-th].
- [30] A. De Felice, R. Kase and S. Tsujikawa, Phys. Rev. D **83** (2011) 043515 [arXiv:1011.6132 [astro-ph.CO]].
- [31] S. Nesseris, A. De Felice and S. Tsujikawa, Phys. Rev. D **82** (2010) 124054 [arXiv:1010.0407 [astro-ph.CO]].
- [32] A. De Felice and S. Tsujikawa, Phys. Rev. Lett. **105** (2010) 111301 [arXiv:1007.2700 [astro-ph.CO]].
- [33] A. Ali, R. Gannouji and M. Sami, Phys. Rev. D **82** (2010) 103015 [arXiv:1008.1588 [astro-ph.CO]].
- [34] T. Kobayashi, M. Yamaguchi and J. Yokoyama, Phys. Rev. Lett. **105** (2010) 231302 [arXiv:1008.0603 [hep-th]].
- [35] C. Burrage, C. de Rham, D. Seery and A. J. Tolley, JCAP **1101** (2011) 014 [arXiv:1009.2497 [hep-th]].
- [36] T. Kobayashi, M. Yamaguchi and J. Yokoyama, Prog. Theor. Phys. **126** (2011) 511 [arXiv:1105.5723 [hep-th]].
- [37] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B **485** (2000) 208 [hep-th/0005016].
- [38] C. Deffayet, Phys. Lett. B **502** (2001) 199 [hep-th/0010186].
- [39] C. Deffayet, G. R. Dvali and G. Gabadadze, Phys. Rev. D **65** (2002) 044023 [astro-ph/0105068].
- [40] E. Babichev, C. Deffayet and G. Esposito-Farese, Phys. Rev. Lett. **107** (2011) 251102 [arXiv:1107.1569 [gr-qc]].
- [41] R. Kimura, T. Kobayashi and K. Yamamoto, Phys. Rev. D **85** (2012) 024023 [arXiv:1111.6749 [astro-ph.CO]].
- [42] N. Chow and J. Khoury, Phys. Rev. D **80** (2009) 024037 [arXiv:0905.1325 [hep-th]].
- [43] G. Gabadadze and A. Iglesias, Phys. Lett. B **639** (2006) 88 [hep-th/0603199].
- [44] G. Gubitosi, E.V. Linder, Phys. Lett. B 703, 113 (2011) [arXiv:1106.2815]
- [45] C. de Rham, L. Heisenberg, Phys. Rev. D 84, 043503 (2011) [arXiv:1106.3312]
- [46] K. van Acoleyen, J. van Doorsselaere, Phys. Rev. D 83, 084025 (2011) [arXiv:1102.0487]
- [47] C. Germani, L. Martucci and P. Moyassari, arXiv:1108.1406 [hep-th].
- [48] S.F. Daniel, E.V. Linder, Phys. Rev. D 82, 103523 (2010) [arXiv:1008.0397]
- [49] Y-S. Song, L. Hollenstein, G. Caldera-Cabral, K. Koyama, JCAP 1004, 018 (2010) [arXiv:1001.0969]
- [50] E.V. Linder, Phil. Trans. Roy. Soc. A 369, 4985 (2011) [arXiv:1103.0282]
- [51] E. Bertschinger, Phil. Trans. Roy. Soc. A 369, 494 (2011) [arXiv:1111.4659]
- [52] J. Dossett, J. Moldenhauer and M. Ishak, Phys. Rev. D **84** (2011) 023012 [arXiv:1103.1195 [astro-ph.CO]].
- [53] R. Kimura and K. Yamamoto, arXiv:1112.4284 [astro-ph.CO].
- [54] M. A. Luty, M. Porrati and R. Rattazzi, JHEP **0309** (2003) 029 [hep-th/0303116].
- [55] C. Charmousis, R. Gregory, N. Kaloper and A. Padilla, JHEP **0610** (2006) 066 [hep-th/0604086].
- [56] F. Berkhahn, D. D. Dietrich and S. Hofmann, JCAP **1011** (2010) 018 [arXiv:1008.0644 [hep-th]].
- [57] A. De Felice and S. Tsujikawa, arXiv:1110.3878 [gr-qc].
- [58] E. Babichev, V. Mukhanov and A. Vikman, JHEP **0802** (2008) 101 [arXiv:0708.0561 [hep-th]].